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近年 **MRI** の技術が確立し、生体の内部の様子が画像として表示できる様になった。MRI は体内の各部位における水素密度を測定する技術である。しかし**初期の癌**ではその水素密度は正常な部位と全く変わりなく、ただその堅さ（即ちずれ弾性率）だけが異なる事が知られている。つまり MRI をもってしても初期癌の発見は理論的に難しい。しかしこの堅さの相違は大きいので、もしも医師が体内にまでその手を伸ばし触診するような**仮想的な触診**が可能になれば、初期癌を発見することが大変容易になる。この仮想的触診を、実現する可能性が高い有力な方法として **MRE 法**(magnetic resonance elastography)がある。これは 1995 年頃に R. Mathupillai とその研究協力者達により提唱された方法で、「身体表面に振動を与えて、その振動が波（**弾性波**）として伝わる様子を MRI 装置によって観測すれば、堅さがわかるのではないか」という考え方に基づく

MRE 法による初期癌診断法は、X 線被爆の危険がないより安全で確実な方法となり得る可能性を持っているので、世界各国の研究者が注目して研究している。外国では例えば、アメリカの A. Manduca 教授達の研究グループ、ドイツの I. Sack 教授達の研究グループなどがある。日本では、研究分担者の**菅幹生**助教授を中心に研究が開始された。各国における MRE 法による初期癌診断法は、いずれの研究もまだ初期癌のファントムを使った MRE 法による検査・診断法の研究が始まった段階であり、いろいろな研究結果が報告されているが、理論的研究に裏打ちされた信頼性の高い研究結果が無いのが現状である。その最大の理由は、次の**問題点**にある。即ち、(i)初期癌のファントム同定逆問題の MRE 法による計測データの数値実験による忠実な再現が行われてこなかったこと、(ii)この逆問題を解く有効な数学的方法が定まらないことにある。

本研究では、上記の二つの問題点の解決を目指すことを目標とした。問題点(i)は、京都大学の**藤原宏志**助手の協力を得て、数値解析と数値実験を行い、菅幹生が初期癌のファントムに対して MRE 法で計測したデータを、数値実験で忠実に再現した。

問題点(ii)の解決方法の第一段階として、ロバストな（即ち誤差に強い）数値微分法の開発とその適用を試みた。数値微分法とは、m 回連続微分可能な未知関数の有限個の点における近似値（以下**観測データ**とよぶ）が与えられたとき、それから未知関数を近似的に復元したり、その未知関数が m＋１回微分可能でない点を見つけたりする逆問題に対する解法のことである。本研究で開発した数値微分法は、Tikhonov の正則化による**最小二乗法**に基づく、簡便で高精度・ロバストな方法である。その詳細については、論文[1]と論文 draft[2]を参照されたい。

この数値微分法を次のように利用すると、人体の部位の仮想的触診が可能である。今、観測データと

して、MRE 法で計測した周波数 k の横波の変位ベクトルの一成分 u の有限個の点（以下**計測点**　とよぶ）における値をとる。数値微分法により u とそのラプラス作用素Δによる微分Δu を近似的に求め、u が満たす Helmholtz 型方程式：$(\rho k^2 + \mu \Delta)u=0$　を使って、弾性係数の一つ**せん断率（＊）**$\mu = \rho k^2 u/(\Delta u)$（堅さを表す物理定数）を近似的に求めることが可能である。但し、この議論は弾性係数が均質な所でのみ可能である。弾性係数が不連続となる初期癌のファントムの境界の同定は、Δu の不連続点としてとらえることが出来るので、数値微分法による不連続点の同定法を使って同定する。従って、まず初期癌のファントムの境界を同定し、次にこの境界から離れたところで公式(*)を用いてμを求める。この値が大きければ大きいほど硬いことを表しているので、このようにして人体の部位の硬さが分かる。

次にこの方法の有効性を数値実験により検証する。実際の数値実験を行うに当たっては、藤原宏志による菅幹生の実験結果の数値的再現で得られた横波の変位ベクトルの一成分関数 u を, MRE 法で計測される u の代わりに用い、この u の有限個の計測点における値に誤差をランダムに加えて観測データとする。観測データから数値微分法により u を近似的に求め、Δu の値が相対的に十分大きくなる計測点を求める。これがファントムの境界を近似的に与えるはずである。次にファントムの内部で数値微分法によりΔu を近似的に求め、公式（＊）を用いればせん断率μ（以下**逆問題の解**とよぶ）が近似的に求まり、これが周辺の部位に比べて大きければ、確かに癌のファントムと診断できる。この癌のファントム同定逆問題に対して数値微分法の有効性を実証するには、観測データの誤差が **2 パーセント**のとき、**逆問題の解の誤差**が **10 パーセント以内**である必要がある（以下この要請を**誤差拡大抑制**とよぶ)。癌のファントム同定逆問題ではないが、簡単な幾つかの数値例題に対して本研究開発の数値微分法が、誤差拡大抑制であることを確認した。

ところで、菅幹生の実験で得られるせん断係数は、周波数依存性を持つ。これは初期癌ファントム同定逆問題の基礎方程式が、非圧縮定常弾性波動方程式ではなく、何らかの**粘弾性**を有する方程式と思われる。そこで非圧縮定常粘弾性波動方程式を基礎方程式にとり、菅幹生の実験結果の前よりもより忠実な数値的再現とその数値データを観測データとしてせん断率と粘性率を求める方法について研究した。得られた研究成果は、次の通りである。基礎方程式を物質係数が非均質な２次元非圧縮定常弾性波動方程式或いは非圧縮定常粘弾性波動方程式とした場合にいわゆる oscillating-decaying solution、Runge の近似定理、楕円型方程式の Cauchy 問題の解法、双曲型方程式の初期値問題の解法を用いて、せん断率或いはせん断率と粘性率を求める方法を与えた。(詳細については論文[3]を参照されたい。)　前述の物質係数が区分的に均質な場合の問題点(ii)に対する第一段階の解決方法に比べると、はるかに複雑な解決方法になってしまっているが、数学的には厳密な解決方法を与えている。また藤原宏志の協力を得て、基礎方程式を区分的に均質な２次元非圧縮定常粘弾性波動方程式とした場合に、菅幹生の実験結果を前よりもより忠実に再現した。そして均質な所で有効な（＊）と類似な公式よりせん断率と粘性率を求め、せん断率に関しては誤差拡大抑制も満足されることを示した。そのより詳細[4]は、藤原宏のホームページよりダウンロードできる。

文献

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Numerical differentiation for the second order derivatives of functions with two variables

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Abstract

A regularized optimization problem for computing numerical differentiation for the second order derivative of functions with two variables from noisy values at the scattered points is discussed in this article. The authors give the proof of the existence and uniqueness of the solution to this problem, the construction scheme of the solution is based on bi-harmonic Green function and the convergence estimate of the regularized solution to the exact solution for the problem under a simple choice of regularization parameter. The efficiency of the reconstruction scheme is shown by some numerical examples.

Key words: numerical differentiation, Tikhonov regularization, Green function
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1 Introduction

Numerical differentiation is a problem to determine the derivatives of an unknown function from the given noisy values of the unknown function at the scattered points. Hereafter, for simplicity, we abbreviate this determination of derivatives by *numerical differentiation from noisy scattered data.*It arises from many scientific researches and applications, but it is an ill-posed problem,

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which means, the small errors in the measurement of the function may lead to large errors in its computed derivatives ([7], [9], [14]). There have been many approaches proposed ([8], [9],[12]) for treating the numerical differentiation problem.

A. G. Ramm proposed an approach for getting stable numerical differentiation by using finite-difference methods in 1968[12]. Recently A. G. Ramm and A. B. Smirnova gave the error estimate and presented some numerical examples for this approach [13]. Their detailed study was given basically for functions with one variable. The error estimate of this method is precise and optimal. This numerical differentiation based on the finite-difference methods only gives piecewise constant functions which lacks the smoothness.

On the other hand, M. Hanke and O. Scherzer proposed another approach to the problem based on discrepancy principle for the least square method combined with Tikhonov regularization finding the minimizer in natural cubic splines [9]. They only considered it for functions with one variable. Y. B. Wang et al. adopted the idea given in [9] to treat irregular grid and gave a simple way to choose the regularization parameter. The byproduct of this method is that it can identify the discontinuity of an unknown function from the noisy values of the unknown function at the scattered points. This identification of the discontinuity was used to find discontinuous solutions of Abel integral equations [2] and edge detection of image [10]. The numerical results showed that this method was quite efficient.

The higher order and two dimensional numerical differentiation along the line of this method were given in [15] and [17], respectively. For the two dimensional case, the new ingredient was that the variational problem for the regularized minimization problem is solved using the Green function for the Laplacian with Dirichlet boundary condition and a scheme for computing the first order derivative was given in [17]. The numerical example showed that this method was efficient. But in many applications, it is necessary to compute higher order derivatives, for example, in the plate bending problem, the bending moments are obtained from the second derivatives of the vertical displacement of the plate [1], so in this paper we will give a numerical differentiation of the second order derivatives from noisy scattered data. The error estimate for our method has the form and order similar to that of [13] in terms of the noise level of the data and size of the irregular grid. But the order in terms of the size of irregular grid is slightly worse than that of [13]. Comparing these two error estimates of [13] and ours, we have to be aware of the difference that the former is for functions with one variable and the latter is for functions with two variables.

The paper is organized as follows: in section 2, we describe the problem in detail and prove the existence and uniqueness of the solution; in section 3 we give the convergence estimate of the regularized solution to the exact solution for the problem by error estimate; the numerical examples are given in section 4; in section 5, we discuss the efficiency of this method by analyzing the results of the numerical examples in section 4; we give the algorithm for computing the Green function in Appendix.

2 Problem and some results

Suppose that $\Omega \subset \mathbb{R}^2$ is a bounded domain with C^2 boundary and $\varrho = \varrho(x)$ is a function defined in Ω. Let N be a natural number and $\{x_i\}_{i=1}^N$ be a group of points in Ω. We assume that Ω is divided into N parts $\{\Omega_i\}_{i=1}^N$, and there is only one point of $\{x_i\}_{i=1}^N$ in each part. For simplicity we also assume that the areas $|\Omega_i|$ of all $\Omega_i (1 \leq i \leq N)$ are same. We denote by d_i the diameter of Ω_i and let $d = \max\{d_i\}$.

We will discuss the following problem:

Suppose that we know the approximate value $\tilde{\varrho}_i$ of $\varrho(x)$ at point x_i, i.e.

$$|\tilde{\varrho}_i - \varrho(x^i)| \leq \delta, \qquad i = 1, 2, \cdots, N, \tag{2.1}$$

where $\delta > 0$ is a given constant called the error level.

We want to find a function $f_*(x)$ which approximates function $\varrho(x)$ such that $\|f_* - \varrho\|_{H^2(\Omega)}$ is small and

$$\lim_{d \to 0, \delta \to 0} \|f_* - \varrho\|_{H^2(\Omega)} = 0.$$

Assuming that there are two functions $\phi(x) \in H^{7/2}(\partial\Omega)$ and $\varphi(x) \in H^{3/2}(\partial\Omega)$ satisfying $\|\phi(x) - \varrho(x)\|_{H^{7/2}(\partial\Omega)} \leq \delta$ and $\|\varphi(x) - \triangle\varrho(x)\|_{H^{3/2}(\partial\Omega)} \leq \delta$, we treat this problem as the following optimization problem by using Tikhonov regularization method.

Problem 2.1 *Define a cost functional* $\Phi(f)$*:*

$$\Phi(f) = \frac{1}{N} \sum_{j=1}^{N} (f(x^j) - \tilde{\varrho}_j)^2 + \alpha \|\triangle^2 f\|^2_{L^2(\Omega)}, \quad f \in H$$

where $H = \{f | f \in H^4(\Omega), f|_{\partial\Omega} = \phi, \triangle f|_{\partial\Omega} = \varphi\}$*, and* $\alpha > 0$ *is a regularization parameter. Then, the problem is then to find* $f_* \in H$ *such that* $\Phi(f_*) \leq \Phi(f)$ *for every* $f \in H$*.*

Then we will prove the existence and uniqueness of the minimizer of Problem 2.1.

Theorem 2.2 *Suppose that $f_* \in H$ is the solution of the following variational problem:*

$$\int_\Omega \triangle^2 f \triangle^2 h dx = -\frac{1}{\alpha N}\sum_{j=1}^{N}(f(x^j) - \tilde{\varrho}_j)h(x^j) \tag{2.2}$$

for all $h \in \hat{H} = \{h | h \in H^4(\Omega), h|_{\partial\Omega} = \triangle h|_{\partial\Omega} = 0\}$. Then f_* is the minimizer of Problem 2.1. Moreover, the minimizer of Problem 2.1 is unique.

Remark 2.3 *We will prove the existence of a solution of* (2.2) *later in Theorem 2.4.*

Proof of Theorem 2.2. For any $f \in H$, let $h = f - f_*$, then $h \mid_{\partial\Omega} = 0$ and $\triangle h \mid_{\partial\Omega} = 0$. It is easy to have the following equations:

$$\begin{aligned}\Phi(f) - \Phi(f_*) =& \frac{1}{N}\sum_{j=1}^{N}(f(x^j) - f_*(x^j))(f(x^j) + f_*(x^j) - 2\tilde{\varrho}_j) \\ &+\alpha \int_\Omega \Big[(\triangle^2 f)^2 - (\triangle^2 f_*)^2\Big] dx \\ =& I_1 + \alpha I_2\end{aligned} \tag{2.3}$$

and

$$I_1 = \frac{1}{N}\sum_{j=1}^{N} 2(f_*(x^j) - \tilde{\varrho}_j)h(x^j) + h^2(x^j)\,.$$

By the definition of f_*, we have

$$\begin{aligned}I_2 =& \int_\Omega [(\triangle^2 f)^2 - (\triangle^2 f_*)^2]dx = \|\triangle^2 h\|^2_{L^2(\Omega)} + 2\int_\Omega \triangle^2 h \cdot \triangle^2 f_* dx \\ =& \|\triangle^2 h\|^2_{L^2(\Omega)} - \frac{2}{\alpha N}\sum_{j=1}^{N}(f_*(x^j) - \tilde{\varrho}_j)h(x^j)\,.\end{aligned}$$

Substituting the equations I_1 and I_2 into (2.3) gives

$$\Phi(f) - \Phi(f_*) = \frac{1}{N}\sum_{j=1}^{N} h^2(x^j) + \alpha\|\triangle^2 f - \triangle^2 f_*\|^2_{L^2(\Omega)} \geq 0\,.$$

Thus, f_* is a minimizer of Problem 2.1.

If there is another $f^* \in H$ minimizing Problem 2.1, denote $g = f^* - f_*$, then function g satisfies: $\int_\Omega (\triangle^2 g)^2 dx = 0$ and $g|_{\partial\Omega} = 0, \triangle g|_{\partial\Omega} = 0$. Hence, $g(x) \equiv 0$ for $x \in \Omega$. So $f^* = f_*$. Therefore, the uniqueness of the minimizer of Problem 2.1 has been proven. □

To solve numerical differentiation problem, it is necessary to provide a scheme for constructing f_*. For that, by an a priori argument using Green function of bi-harmonic operator, we construct f_*. It will be shown as a theorem that the constructed f_* is the solution of (2.2).

Let's recall the definition of the bi-harmonic Green function before going into the construction. Function $G(x, y)$ with fixed $y \in \Omega$ is called the bi-harmonic Green function if it satisfies

$$\triangle_x^2 G(x, y) = \delta(x - y) \quad \text{in } \Omega$$

$$G|_{\partial\Omega} = 0, \quad \triangle_x G|_{\partial\Omega} = 0\,.$$

We can obtain $G(x, y)$ by solving

$$\begin{aligned}&\triangle_x F(x, y) = \delta(x - y) \quad \text{in } \Omega\\ &F(x, y)|_{\partial\Omega} = 0\end{aligned}$$

and

$$\begin{aligned}&\triangle_x G(x, y) = F(x, y) \quad \text{in } \Omega\\ &G(x, y)|_{\partial\Omega} = 0\,.\end{aligned}$$

We denote $\triangle_1$ as the Laplacian operator for the first argument, and $\triangle_2$ as the Laplacian operator for the second argument. Since $G(x, y) = G(y, x)$ and $F(x, y) = F(y, x)$ for $x, y \in \Omega$, we have

$$\triangle_2 G(y, x) = \triangle_1 G(x, y) = F(x, y) = F(y, x) = \triangle_1 G(y, x)\,. \tag{2.4}$$

Now we will propose a scheme to obtain the solution of the Eq. (2.2). Taking $h = G(x, y)$ in (2.2) and using the definition of Green function, we obtain

$$\begin{aligned}-\tfrac{1}{N}\textstyle\sum_{j=1}^{N}(f_*(x^j) - \tilde{\varrho}_j)G(x^j, y) &= \int_\Omega \alpha\triangle^2 f_*(x) \cdot \triangle_x^2 G(x, y)dx\\ &= \alpha\triangle^2 f_*(y).\end{aligned}$$

Multiply the two sides of the above equation by $G(x, y)$ and integrate it on Ω, we obtain by integrating by parts

$$
\begin{aligned}
&-\frac{1}{N}\sum_{j=1}^{N}(f_*(x^j)-\tilde{\varrho}_j)\int_\Omega G(x^j,x)G(x,y)dx \\
=&\,\alpha\int_\Omega \triangle^2 f_*(x)\cdot G(x,y)dx \\
=&\,\alpha\int_{\partial\Omega}(\frac{\partial}{\partial\nu}\triangle f_*(x)\cdot G(x,y)-\frac{\partial}{\partial\nu}G(x,y)\cdot\triangle f_*(x))ds(x) \\
&+\alpha\int_\Omega \triangle f_*(x)\cdot\triangle_x G(x,y)dx \\
=&-\alpha\int_{\partial\Omega}\frac{\partial}{\partial\nu}G(x,y)\cdot\varphi(x)ds(x)-\alpha\int_{\partial\Omega}\frac{\partial}{\partial\nu}\triangle_x G(x,y)\cdot\phi(x)ds(x) \\
&+\alpha f_*(y)\,,
\end{aligned}
$$

where ν is the unit normal of $\partial\Omega$ directed outside Ω. Rewrite the above equation in the form:

$$
\begin{aligned}
&\alpha f_*(x)+\frac{1}{N}\sum_{j=1}^{N}(f_*(x^j)-\tilde{\varrho}_j)\int_\Omega G(x^j,y)G(y,x)dy \qquad (2.5)\\
=&\,\alpha\int_{\partial\Omega}\frac{\partial}{\partial\nu}G(y,x)\cdot\varphi(y)ds(y)+\alpha\int_{\partial\Omega}\frac{\partial}{\partial\nu}\triangle_y G(y,x)\cdot\phi(y)ds(y)\,.
\end{aligned}
$$

By defining

$$a_j(x)=\int_\Omega G(x^j,y)G(y,x)dy, \qquad (2.6)$$

$$b(x)=\int_{\partial\Omega}\frac{\partial}{\partial\nu}\triangle_y G(y,x)\cdot\phi(y)ds(y)+\int_{\partial\Omega}\frac{\partial}{\partial\nu}G(y,x)\cdot\varphi(y)ds(y) \qquad (2.7)$$

and

$$c_j=-\frac{1}{\alpha N}(f_*(x^j)-\tilde{\varrho}_j) \qquad (2.8)$$

(2.5) becomes

$$f_*(x)=\sum_{j=1}^{N}c_j a_j(x)+b(x)\,. \qquad (2.9)$$

Now the problem of constructing f_* reduces to computing the coefficients c_j from $\tilde{\varrho}_j$, $\varphi(x)$ and $\phi(x)$. From (2.8) and (2.9) we obtain

$$c_j=-\frac{1}{\alpha N}(f_*(x^j)-\tilde{\varrho}_j)=-\frac{1}{\alpha N}(\sum_{k=1}^{N}a_k(x^j)c_k+b(x^j)-\tilde{\varrho}_j)\,. \qquad (2.10)$$

Let

$$
\mathbf{A}=\begin{pmatrix}
\alpha N+a_1(x^1) & a_2(x^1) & a_3(x^1) & \cdots & a_N(x^1)\\
a_1(x^2) & \alpha N+a_2(x^2) & a_3(x^2) & \cdots & a_N(x^2)\\
\cdots & \cdots & \cdots & \cdots & \cdots\\
a_1(x^N) & a_2(x^N) & a_3(x^N) & \cdots & \alpha N+a_N(x^N)
\end{pmatrix}
$$

and

$$\mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \\ \cdots \\ c_N \end{pmatrix}, \mathbf{b} = \begin{pmatrix} \tilde{\rho_1} - b(x^1) \\ \tilde{\rho_2} - b(x^2) \\ \cdots \\ \tilde{\rho_N} - b(x^N) \,. \end{pmatrix},$$

Then (2.10) becomes the linear equations $\mathbf{Ac} = \mathbf{b}$. Solving this equations, we will obtain coefficients c_j, which finishes the construction of f_*.

Theorem 2.4 *Suppose function $f_* = \sum_{j=1}^N c_j a_j(x) + b(x)$ where $a_j(x)$ and $b(x)$ are defined in* (2.6) *and* (2.7)*, $\{c_j\}_{j=1}^N$ is the solution of linear system* (2.10)*, then f_* is the solution of* (2.2)*.*

Proof. For every $x \in \partial\Omega$, from the definition of Green function, we know that $G(x, y) = G(y, x) = 0$ for $y \in \Omega$. So

$$a_j(x) = \int_\Omega G(x^j, y) \cdot G(y, x) dy = 0.$$

Assume that $\hat{\phi} \in H^2(\Omega)$ is an extension of ϕ to Ω and $\hat{\varphi} \in H^2(\Omega)$ is an extension of φ over Ω, then integrating by parts yields

$$b(x) = \phi(x) \quad (x \in \partial\Omega).$$

Thus we have $f_*(x)|_{\partial\Omega} = \phi(x)$.

We also have

$$\triangle a_j(x) = \int_\Omega G(x^j, y) \triangle_x G(x, y) dy = 0 \qquad (x \in \partial\Omega)$$

Since

$$\begin{aligned} b(x) = & \int_\Omega \hat{\phi}(y) \triangle_y^2 G(y, x) dy - \int_\Omega \triangle_y G(y, x) \triangle \hat{\phi}(y) dy \\ & + \int_\Omega \hat{\varphi}(y) \triangle_y G(y, x) dy - \int_\Omega G(y, x) \triangle \hat{\varphi}(y) dy \quad (x \in \partial\Omega), \end{aligned}$$

then we will have using the definition of $F(x, y)$ and (2.4),

$$\begin{aligned} \triangle b(x) = & \triangle \hat{\phi}(x) - \int_\Omega \triangle_x (\triangle_y G(y, x)) \triangle \hat{\phi}(y) dy \\ & + \int_\Omega \hat{\varphi}(y) \triangle_x (\triangle_y G(y, x)) dy - \int_\Omega \triangle_x G(x, y) \triangle \hat{\varphi}(y) dy \\ = & \triangle \hat{\phi}(x) - \int_\Omega \triangle_x^2 G(x, y) \triangle \hat{\phi}(y) dy + \int_\Omega \hat{\varphi}(y) \triangle_x^2 G(x, y)) dy \\ = & \varphi(x) \qquad (x \in \partial\Omega). \end{aligned}$$

Thus we have $\triangle f_*(x)|_{\partial\Omega} = \varphi(x)$.

Moreover, from the definition of $a_j(x)$ and $b(x)$, we know that for every $x \in \Omega$

$$\triangle^2 a_j(x) = \int_\Omega G(x^j, y)\triangle_x^2 G(x, y)dy = G(x^j, x) \tag{2.11}$$

and

$$\triangle^2 b(x) = \triangle\hat{\varphi}(x) - \int_\Omega \triangle_x^2 G(x, y)\triangle\hat{\varphi}(y)dy = \triangle\hat{\varphi}(x) - \triangle\hat{\varphi}(x) = 0. \tag{2.12}$$

Since $G(x^j, x) \in L^2(\Omega)$, so we have $\triangle^2 f_*(x) \in L^2(\Omega)$. From the well-posedness of the Possion equation with inhomogeneous Dirichlet boundary condition, we know $f_* \in H^4(\Omega)$. Furthermore $f_* \in H$.

For any $h \in \hat{H}$, we have

$$\begin{aligned}
&\int_\Omega \triangle^2 f_* \triangle^2 h dx = \int_\Omega \sum_{j=1}^N c_j G(x^j, x)\triangle^2 h(x)dx \\
=&\sum_{j=1}^N c_j \left(\int_\Omega \triangle G(x^j, x)\triangle h(x)dx - \int_{\partial\Omega} \triangle h(x)\frac{\partial G(x^j, x)}{\partial\nu}ds(x) \right. \\
&\left. + \int_{\partial\Omega} G(x^j, x)\frac{\partial\triangle h(x)}{\partial\nu}ds(x) \right) \\
=&\sum_{j=1}^N c_j \left(\int_\Omega \triangle^2 G(x^j, x)h(x)dx - \int_{\partial\Omega} h(x)\frac{\partial\triangle G(x^j, x)}{\partial\nu}ds(x) \right. \\
&\left. + \int_{\partial\Omega} \triangle G(x^j, x)\frac{\partial h(x)}{\partial\nu}ds(x) \right) \\
=&\sum_{j=1}^N c_j h(x^j) = -\frac{1}{\alpha N}\sum_{j=1}^N (f_*(x^j) - \tilde{\varrho}_j)h(x^j).
\end{aligned}$$

So f_* is the solution of (2.2). This completes the proof. □

The solution of the linear equations exists and is unique, because if we assume $\tilde{\varrho}_i = 0, i = 1, \cdots, N$, and $\phi(x) = \varphi(x) = 0$, then we know that there is only one minimizer of Problem 2.1 , which is $f_*(x) \equiv 0, x \in \Omega$. It is obvious $\mathbf{c} = 0$ is a solution of $\mathbf{Ac} = 0$ and if there is another $\hat{\mathbf{c}}$ satisfying $\mathbf{A}\hat{\mathbf{c}} = \mathbf{0}$, then we will have a function $\hat{f} \neq 0$ which is also a minimizer of Problem 2.1. This is a contradiction so the homogenous linear equations only has a trivial solution. Thus the solution of the linear equations exists and is unique.

3 Error estimate

In this section we will give a convergence estimate for our proposed solution under a priori choice of the regularization parameter. The proof will use the following lemma.

Lemma 3.1 *Let E be a bounded domain in $\mathbb{R}^n$ with Lipschitz boundary ∂E, $u \in W^{1,p}(E)$, and suppose that $n < p \leq \infty$, then*

$$|u(x) - u(y)| \leq K|x - y|^{1-\frac{n}{p}} \|u\|_{1,p,\Omega}$$

where K is independent of u.

This lemma can be seen in page 27 of [6].

According to the result of [3], we choose the regularization parameter $\alpha = \delta^2$. Such choice has been proven quite effective (see [16]). We give the error estimate in the following theorem:

Theorem 3.2 *Suppose f_* is the minimizer of Problem 2.1 and $\varrho \in H^4(\Omega)$. Let $e = f_* - \varrho$ and choose $\alpha = \delta^2$, then we have the following error estimate*

$$\|\triangle e\|_{L^2(\Omega)} \leq L_1 d^{\frac{1}{2}} + L_2\delta^{\frac{1}{2}}, \qquad \|e\|_{L^2(\Omega)} \leq L_3 d + L_4\delta$$
$$\|\nabla e\|_{L^2(\Omega)} \leq L_5 d^{\frac{3}{4}} + L_6\delta^{\frac{1}{2}}, \qquad \|\nabla\triangle e\|_{L^2(\Omega)} \leq L_7 d^{\frac{1}{4}} + L_8\delta^{\frac{1}{4}}$$

where L_i are constants which depend on Ω, $\|\phi\|_{H^{7/2}(\partial\Omega)}$, $\|\varphi\|_{H^{3/2}(\partial\Omega)}$ and $\|\triangle^2\varrho\|_{L^2(\Omega)}$.

Proof. For simplicity, we use the abbreviation $\|\cdot\|_{L^2} = \|\cdot\|_{L^2(\Omega)}$. Since $\delta^2\|\triangle^2 f_*\|^2_{L^2} \leq \Phi(f_*) \leq \Phi(\varrho) \leq \delta^2 + \delta^2\|\triangle^2\varrho\|^2_{L^2}$, it is easy to see that $\|\triangle^2 e\|_{L^2} \leq 1 + 2\|\triangle^2\varrho\|_{L^2}$. Also, from the well-posedness of the boundary value problem

$$\begin{cases} \triangle^2 e = g \quad \text{in} \quad \Omega \\ e = k, \quad \triangle e = \ell \quad \text{on} \quad \partial\Omega \end{cases} \tag{3.1}$$

with given $g \in L^2(\Omega)$, $k \in H^{7/2}(\partial\Omega)$, $\ell \in H^{3/2}(\partial\Omega)$ and the continuity of the trace operator, there are constants C_1 and C_2 such that $\|\frac{\partial}{\partial\nu}e\|_{L^2(\partial\Omega)} \leq C_1$ and $\|\frac{\partial}{\partial\nu}\triangle e\|_{L^2(\partial\Omega)} \leq C_2$. Hereafter, C_i's are general constants which may depend on Ω, $\|\phi\|_{H^{7/2}(\partial\Omega)}$, $\|\varphi\|_{H^{3/2}(\partial\Omega)}$ and $\|\triangle^2\varrho\|_{L^2}$.

So, by $\|e\|_{L^2} \leq \delta$, $\|\triangle e\|_{L^2} \leq \delta$,

$$
\begin{aligned}
\|\triangle e\|_{L^2}^2 &= \int_\Omega |\triangle e|^2 dx \\
&= \int_{\partial\Omega} \triangle e \cdot \frac{\partial}{\partial \nu} e dS - \int_{\partial\Omega} e \cdot \frac{\partial}{\partial \nu} \triangle e dS + \int_\Omega \triangle^2 e \cdot e dx \\
&\leq \|e\|_{L^2} \cdot \|\triangle^2 e\|_{L^2} + C_1\delta + C_2\delta.
\end{aligned}
$$

Here, note that the general constants C_1, C_2 can be different in each estimate. We rewrite $\|e\|_{L^2}$ as

$$
\begin{aligned}
\|e\|_{L^2}^2 &= \int_\Omega e^2(x)dx = \sum_{i=1}^{N} \int_{\Omega_i} e^2(x)dx \\
&= \sum_{i=1}^{N} \int_{\Omega_i} e(x)(e(x) - e(x^i))dx + \sum_{i=1}^{N} \int_{\Omega_i} e(x^i)(e(x) - e(x^i))dx \\
&\quad + \sum_{i=1}^{N} \int_{\Omega_i} e^2(x^i)dx \\
&= I_3 + I_4 + I_5.
\end{aligned}
$$

Now we estimate I_3, I_4, and I_5.

$$
\begin{aligned}
I_3 &= \sum_{i=1}^{N} \int_{\Omega_i} e(x)(e(x) - e(x^i))dx \leq \sum_{i=1}^{N} \int_{\Omega_i} |e(x)||(e(x) - e(x^i))|dx \\
&\leq \sum_{i=1}^{N} \int_{\Omega_i} C_1 |x - x^i|^{1-\frac{n}{p}} \|e\|_{1,p} |e(x)|dx \leq d^{1-\frac{n}{p}} C_1 \|e\|_{1,p} \int_\Omega |e(x)|dx \\
&\leq d^{1-\frac{n}{p}} C_1 \|e\|_{1,p} \|e\|_{L^2} (|\Omega|)^{\frac{1}{2}}
\end{aligned}
$$

where $|\Omega|$ is the area of Ω. The second inequality is obtained from Lemma 3.1 with $n = 2$. We may set $p = 4$, then

$$
I_3 \leq d^{\frac{1}{2}} C_1 (|\Omega|)^{\frac{1}{2}} \|e\|_{1,4} \|e\|_{L^2}.
$$

From the imbedding theorem of Soblev spaces we know that $W^{2,2}(\Omega) \to W^{1,4}(\Omega)$, which means, there is a constant C_1 independent of e satisfying $\|e\|_{1,4} \leq C_1 \|e\|_{2,2}$. By the well-posedness of the boundary value problem for the Possion equation with inhomogeneous Dirichlet condition,

$$
\|e\|_{2,2} \leq C_1 \|\triangle e\|_{L^2} + C_2 \delta.
$$

Hence, we have $I_3 \leq C_1 d^{\frac{1}{2}} \|e\|_{L^2} (\|\triangle e\|_{L^2} + \delta)$.

By the same way, we have

$$
\begin{aligned}
I_4 &= \sum_{i=1}^{N} \int_{\Omega_i} e(x^i)(e(x) - e(x^i))dx \leq \sum_{i=1}^{N} \int_{\Omega_i} |e(x^i)||(e(x) - e(x^i))|dx \\
&\leq d^{\frac{1}{2}} C_1 \|e\|_{1,4} \sum_{i=1}^{N} (\int_{\Omega_i} |e(x^i)|dx) = d^{\frac{1}{2}} C_1 \|e\|_{1,4} \frac{|\Omega|}{N} \sum_{i=1}^{N} |e(x^i)|.
\end{aligned}
$$

Since $\Phi(f_*) \leq \Phi(\varrho)$, we have

$$\frac{1}{N}\sum_{i=1}^{N}(f_*(x^i)-\tilde{\varrho}_i)^2 \leq \delta^2(1+\|\triangle^2\varrho\|^2).$$

So

$$\begin{aligned}
\frac{1}{N}\sum_{i=1}^{N}|e(x^i)| &\leq \frac{1}{N}\sum_{i=1}^{N}(|f_*(x^i)-\tilde{\varrho}_i|+|\tilde{\varrho}_i-\varrho(x^i)|) \\
&\leq \sqrt{\frac{1}{N}\sum_{i=1}^{N}|f_*(x^i)-\tilde{\varrho}_i|^2}+\delta \\
&\leq \delta(\sqrt{1+\|\triangle^2\varrho\|^2}+1).
\end{aligned}$$

Hence, we have $I_4 \leq C_1 d^{\frac{1}{2}}\delta(\|\triangle e\|+\delta)$.

The estimate of I_5 is simple. In fact

$$\begin{aligned}
I_5 &= \sum_{i=1}^{N}\int_{\Omega_i} e^2(x^i)dx = \sum_{i=1}^{N} e^2(x^i)\int_{\Omega_i} dx \leq \frac{1}{N}|\Omega|\cdot\sum_{i=1}^{N} e^2(x^i) \\
&\leq \frac{2}{N}|\Omega|\cdot\sum_{i=1}^{N}((f_*(x^i)-\tilde{\varrho}_i)^2+(\tilde{\varrho}_i-\varrho(x^i))^2) \\
&\leq 2|\Omega|\delta^2(2+\|\triangle^2\varrho\|^2) = C_1\delta^2.
\end{aligned}$$

From all the estimate for I_3 to I_5, we can conclude that

$$\|e\|_{L^2}^2 \leq C_1 d^{\frac{1}{2}}\|e\|_{L^2}(\|\triangle e\|_{L^2}+\delta)+C_2 d^{\frac{1}{2}}\delta\|\triangle e\|_{L^2}+C_3\delta^2.$$

Then, we have

$$\begin{aligned}
\|e\|_{L^2} &\leq C_1 d^{\frac{1}{2}}(\|\triangle e\|_{L^2}+\delta)+C_2 d^{\frac{1}{4}}\delta^{\frac{1}{2}}\|\triangle e\|_{L^2}+C_3\delta \\
&\leq C_1 d^{\frac{1}{2}}\|\triangle e\|_{L^2}+C_2\delta.
\end{aligned}$$

Here, we have some of the estimates in the theorem:

$$\|\triangle e\|_{L^2} \leq C_1 d^{\frac{1}{2}}+C_2\delta^{\frac{1}{2}}$$

and

$$\|e\|_{L^2} \leq C_1 d+C_2\delta$$

Also, since

$$\begin{aligned}
\|\nabla e\|_{L^2}^2 &= \int_{\Omega}\nabla e\cdot\nabla e dx \\
&= -\int_{\Omega}\triangle e\cdot e dx+\int_{\partial\Omega} e\cdot\frac{\partial}{\partial\nu} e dS \\
&\leq \|e\|_{L^2}\cdot\|\triangle e\|_{L^2}+C_1\delta.
\end{aligned}$$

and

$$\begin{aligned}\|\nabla\triangle e\|_{L^2}^2 &= \int_\Omega \nabla\triangle e\cdot\nabla\triangle e dx\\ &= -\int_\Omega \triangle^2 e\cdot\triangle e dx + \int_{\partial\Omega}\triangle e\cdot\frac{\partial}{\partial\nu}\triangle e dS\\ &\le \|\triangle e\|_{L^2}\cdot\|\triangle^2 e\|_{L^2} + C_1\delta,\end{aligned}$$

we have

$$\|\nabla e\|_{L^2} \le C_1 d^{\frac{3}{4}} + C_2\delta^{\frac{1}{2}}$$

$$\|\nabla\triangle e\|_{L^2} \le C_1 d^{\frac{1}{4}} + C_2\delta^{\frac{1}{4}}\ .$$

This completes the proof. □

Remark 3.3 *In this paper, for the simplicity, we assume that the areas of all Ω_i are same. In the real application, this condition may be not easy to be satisfied. But if we denote $V1 = \max^i\{|\Omega_i|\}$ and $V2 = \min_i\{|\Omega_i|\}$ and let $\frac{V1}{V2}$ is bounded with some constant, then we still have the same error estimate.*

Remark 3.4 *In Theorem 3.2, we used Lemma 3.1 to estimate I_3 in which we chose the parameter p to be 4. Actually we can choose any p satisfying $2 < p < \infty$. And we can still use the imbedding theorem of Sobolev spaces $W^{2,2}(\Omega) \to W^{1,p}(\Omega)$. The result will be*

$$\|\triangle e\|_{L^2} \le L_{1p}\cdot d^{1-\frac{2}{p}} + L_{2p}\delta^{\frac{1}{2}}, \qquad \|e\|_{L^2} \le L_{3p} d^{2-\frac{4}{p}} + L_{4p}\delta$$

$$\|\nabla e\|_{L^2} \le L_{5p} d^{\frac{3}{2}-\frac{3}{p}} + L_{6p}\delta^{\frac{1}{2}}, \qquad \|\nabla\triangle e\|_{L^2} \le L_{7p} d^{\frac{1}{2}-\frac{1}{p}} + L_{8p}\delta^{\frac{1}{4}}$$

where L_{ip} are constants depending on $\|\phi\|_{H^{\frac{7}{2}}(\partial\Omega)}$, $\|\varphi\|_{H^{\frac{3}{2}}(\partial\Omega)}$ and Ω, $\|\triangle^2\varrho\|_{L^2}$ and p. So when we choose a larger p we will get a better convergence rate.

Remark 3.5 *If Ω is a polygon, (3.1) with $k = \ell = 0$ is well-posed (see [6] Theorem 4.4.1.3). Since k and ℓ in (3.1) have extensions $f_* - \varrho \in H^4(\Omega)$ and $\triangle(f_* - \varrho) \in H^2(\Omega)$, we can reduce (3.1) to the case with $k = \ell = 0$. Also, the Green formula and the properties of the Sobolev spaces used in this paper still hold when Ω is a bounded domain with Lipschitz boundary. Therefore, our Theorems hold even in the case Ω is a polygon. Our numerical examples in the next section are given for the case Ω is a rectangle. We utilize this remark for the examples.*

4 Numerical examples

We provide numerical examples in this section.

We compute the Green function G by Fourier series for our construction. The detail of the algorithm for constructing f_* and its second derivatives is presented in Appendix.

Let $\varrho(x_1, x_2)$ be a two variable function given by

$$\varrho(x_1, x_2) = \sin(\pi x_1)\sin(2x_2) \quad (x_1, x_2) \in \Omega,$$

where $\Omega = \{(x_1, x_2) : 0 \leq x_1 \leq 1, 0 \leq x_1 \leq 2\pi\}$. We take $\varrho(x_1, x_2)$ as the unknown function to compute the numerical differentiation of its second order derivatives.

We generate the simulated noisy data as follows:

(1) Decomposite Ω into N elements denoted by $\Omega_i (1 \leq i \leq N)$.
(2) In each element, we choose the center of Ω_i as the grid point and get value $\varrho_i (1 \leq i \leq N)$ at each grid point.
(3) Add some noise δ to ϱ_i, then we get the simulated noisy data $\tilde{\varrho}_i (1 \leq i \leq N)$.

We define a cut-off function $\chi(x) \in C^\infty$

$$\chi(x) = \begin{cases} 1, & x \in \Omega' \\ 0, & x \in \partial\Omega \\ t(0 < t < 1), & x \in \Omega \setminus \overline{\Omega'} \end{cases}$$

with $\text{supp}\chi \subset \Omega$ and multiply the measured data $\tilde{\varrho}(x)$ by $\chi(x)$, then we get the value on grid points in our construction and the boundary value ϕ and φ become

$$\phi(x) = f|_{\partial\Omega} = 0, \quad \varphi(x) = \triangle f|_{\partial\Omega} = 0.$$

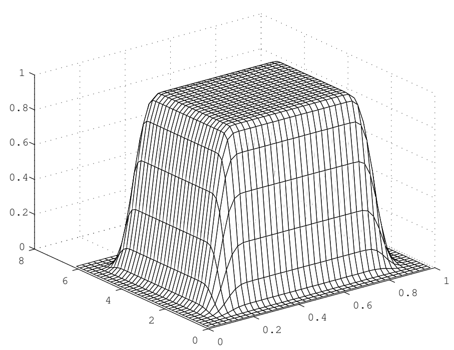

Fig. 1. cut-off function $\chi(x)$

Therefore, we only construct f_*, $\frac{\partial f_*}{\partial x_1 \partial x_1}(x)$ etc. in Ω'. The number of elements in Ω' is denoted by N'.

Fig. 2, 3, 4, 5 illustrate our numerical results for constructing the second order derivatives with $N' = 20^2$, $\delta = 0.01$.

The numerical results about constructing f_*, $f_{*x_1x_1}$, $f_{*x_2x_2}$ and $f_{*x_1x_2}$ are illustrated in Fig. 2 , Fig. 3, Fig. 4 and Fig. 5, respectively. In Fig. 2-5, from the left to right, the three figures correspond to the original function, the constructed function and the constructed error in Ω'.

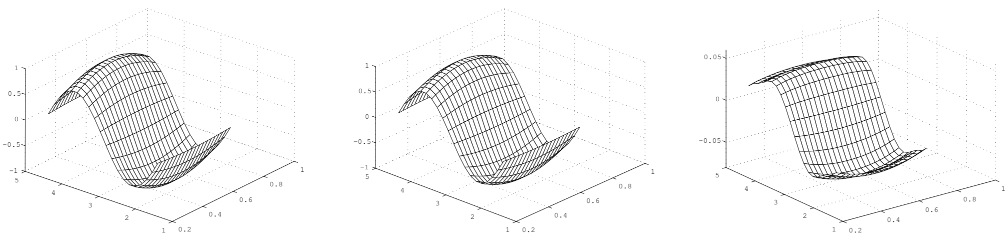

Fig. 2. function ϱ, constructed f_* and constructed error

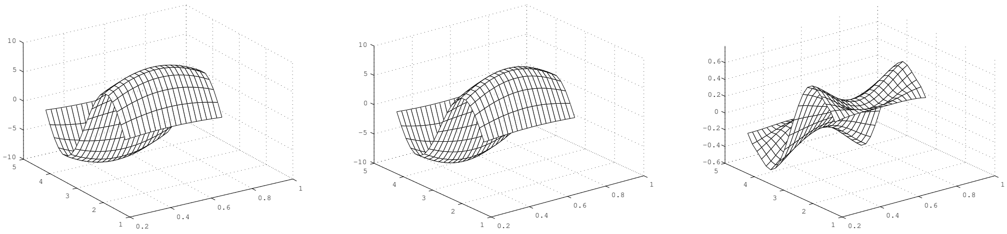

Fig. 3. function $\varrho_{x_1x_1}$, constructed $f_{*x_1x_1}$ and constructed error

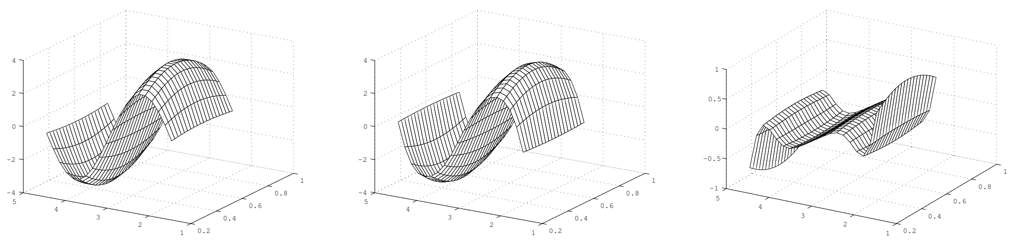

Fig. 4. function $\varrho_{x_2x_2}$, constructed $f_{*x_2x_2}$ and constructed error

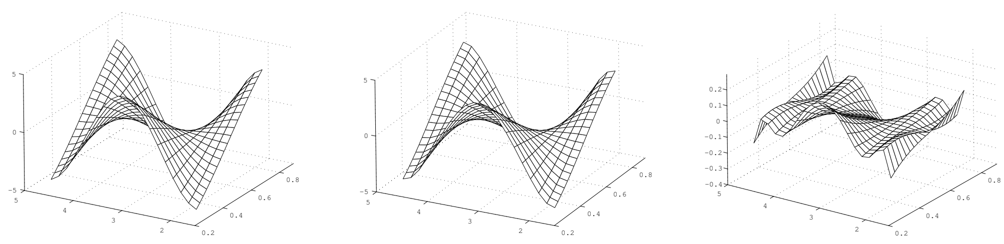

Fig. 5. function $\varrho_{x_1x_2}$, constructed $f_{*x_1x_2}$ and constructed error

Now we investigate how the relative errors depends on N' and δ (see Table 1-Table 5). We define the relative error $\mathcal{E}_{f_*}$, $\mathcal{E}_{f_{*x_1x_1}}$ for constructed f_*, $\frac{\partial f_*}{\partial x_1 \partial x_1}$ by

$$\mathcal{E}_{f_*} = \frac{\left(\sum_{j=1}^{N'}(f_*(x^j) - \varrho(x^j))^2\right)^{1/2}}{\left(\sum_{i=1}^{N}(\varrho(x^j))^2\right)^{1/2}},$$

$$\mathcal{E}_{f_{*x_1x_1}} = \frac{\left(\sum_{j=1}^{N'}(\frac{\partial^2 f_*}{\partial x_1 \partial x_1}(x^j) - \frac{\partial^2 \varrho}{\partial x_1 \partial x_1}(x^j))^2\right)^{1/2}}{\left(\sum_{i=1}^{N}(\varrho(x^j))^2\right)^{1/2}}.$$

$\mathcal{E}_{f_{*x_2x_2}}$, $\mathcal{E}_{f_{*x_1x_2}}$ are defined in the same way.

When δ is increased from 0.005 to 0.05, Table 1 presents the relative errors for contructing f_* and its three second order derivatives with fixed $N' = 20^2$.

Table 2, 3, 4 and 5 present the relative errors when N' being increased from 12^2 to 24^2 with fixed $\delta = 0.005, 0.01$, 0.02, 0.05, respectively.

Table 1
Relative errors(%) with different noise level $\delta(N' = 20^2$, fixed)

δ	$\mathcal{E}_{f_*}$	$\mathcal{E}_{f_{*x_1x_1}}$	$\mathcal{E}_{f_{*x_2x_2}}$	$\mathcal{E}_{f_{*x_1x_2}}$
0.005	4.7612	5.6660	4.0606	1.5112
0.01	6.4743	4.9706	7.5713	1.2932
0.02	11.2376	10.2852	14.0513	4.2602
0.05	32.1106	39.3511	43.7051	21.0809

5 Discussion and conclusion

In Fig. 2, 3, 4, 5, we can observe that the figures for the constructed functions are very similar to that of the corresponding functions. Only by this comparison, we can say our construction is quite good, but the figures are difficult

Table 2
Relative errors(%) with different numbers of grid points $N'(\delta = 0.005$, fixed)

N'	$\mathcal{E}_{f_*}$	$\mathcal{E}_{f_{*x_1x_1}}$	$\mathcal{E}_{f_{*x_2x_2}}$	$\mathcal{E}_{f_{*x_1x_2}}$
12^2	5.7855	4.2359	5.3461	0.9285
16^2	5.1353	4.2500	4.5373	0.8953
20^2	4.7612	5.6660	4.0606	1.5112
24^2	4.5055	7.7803	3.7527	2.3465

Table 3
Relative errors(%) with different numbers of grid points $N'(\delta = 0.01$, fixed)

N'	$\mathcal{E}_{f_*}$	$\mathcal{E}_{f_{*x_1x_1}}$	$\mathcal{E}_{f_{*x_2x_2}}$	$\mathcal{E}_{f_{*x_1x_2}}$
12^2	9.2269	8.1107	11.4623	2.9345
16^2	7.3962	6.0563	9.0640	1.8404
20^2	6.4743	4.9706	7.5713	1.2932
24^2	5.9300	4.3590	6.2529	0.9724

Table 4
Relative errors(%) for different numbers of grid points $N'(\delta = 0.02$, fixed)

N'	$\mathcal{E}_{f_*}$	$\mathcal{E}_{f_{*x_1x_1}}$	$\mathcal{E}_{f_{*x_2x_2}}$	$\mathcal{E}_{f_{*x_1x_2}}$
12^2	19.4912	20.2812	24.1990	10.2773
16^2	14.0671	13.5256	17.3932	6.2511
20^2	11.2376	10.2852	14.0513	4.2602
24^2	9.5989	8.4584	12.0134	3.1269

Table 5
Relative errors(%) for different numbers of grid points $N'(\delta = 0.05$, fixed)

N'	$\mathcal{E}_{f_*}$	$\mathcal{E}_{f_{*x_1x_1}}$	$\mathcal{E}_{f_{*x_2x_2}}$	$\mathcal{E}_{f_{*x_1x_2}}$
12^2	52.5281	93.0336	106.9937	51.5658
16^2	40.4525	56.7802	63.8319	31.0793
20^2	32.1106	39.3511	43.7051	21.0809
24^2	26.3455	29.6180	32.8746	15.3715

to observe the precision of the constructed functions. Therefore, we should investigate the relative errors.

In Table 1, $N' = 20^2$ being fixed, we increase the noise δ from 0.005 to 0.01, the constructed errors $\mathcal{E}_{f_*}$ and $\mathcal{E}_{f_{*x_2x_2}}$ increase, but $\mathcal{E}_{f_{*x_1x_1}}$ and $\mathcal{E}_{f_{*x_1x_2}}$ decrease a little. When the noise is large than 0.01, the constructed errors grow quickly with noise level increasing.

In Table 2, $\delta = 0.005$ being fixed, increasing N' from 12^2 to 24^2, we can see $\mathcal{E}_{f_*}$ and $\mathcal{E}_{f_{*x_1x_1}}$ decrease slowly, whereas $\mathcal{E}_{f_{*x_2x_2}}$ and $\mathcal{E}_{f_{*x_1x_2}}$ increase slowly.

From above phenomenon, we can say when noise level is very small such as less than 0.01, increasing N' cannot alway improve the precision of the constructed functions. Even $\mathcal{E}_{f_{*x_1x_1}}$ and $\mathcal{E}_{f_{*x_1x_2}}$ increase slowly with N' being increased. The reason is that, the numerator in the relative error formula is square summation of the difference between the exact value and the constructed value, when noise is very small, the improvement in constructed functions $f_{*x_1x_1}$ and $f_{*x_1x_2}$

by increasing N' can not compensate the increase of error generated by the number of the summation terms being increased.

In Table 3, $\delta = 0.01$ being fixed, if $N' = 12^2$, the constructed errors are less than 12%. With N' being increased, the constructed errors for all functions become smaller. When $N' = 24^2$, the constructed error $\mathcal{E}_{f_*}$ is 5.9300% and $\mathcal{E}_{f_{*x_1x_1}}$, $\mathcal{E}_{f_{*x_2x_2}}$ and $\mathcal{E}_{f_{*x_1x_2}}$ are less than 7%.

In Table 4 and Table 5, we take $\delta = 0.02$ and $\delta = 0.05$ respectively, and increase N' from $12'$ to $24'$, then the constructed errors for all functions decrease gradually.

From Table 3-Table 5, we conclude that when $\delta \geq 0.01$, increasing N' can improve the precision of numerical differentiation for three second order derivatives of $\varrho(x_1, x_2)$ by our method.

There is the fact that, by using the same noisy data, the constructed error for the mixed derivative is much smaller than that for the other two second derivatives. When N' being increased to 24^2, the constructed errors are about $5 - 6$ times of the noise levels. On the other hand, the constructed error for the mixed second derivative is less than 3 times of the noise level.

These numerical results show that our method is quite efficient for computing numerical differentiation for the second derivatives of functions with two variables from the noisy scattered data. Due to the memory limitation of our computer, we could not take a larger N' to make our numerical results more precise.

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6 Appendix A: Proof of $G(x, y) = G(y, x)$

Here we will give the proof that the solution $G(x, y)$ with fixed $y \in \Omega$ of

$$\begin{cases} \triangle_x^2 G(x,y) = \delta(x-y) & \text{in } \Omega \\ G(x,y)|_{\partial\Omega} = \triangle_x G(x,y)|_{\partial\Omega} = 0 \end{cases}$$

satisfies $G(x, y) = G(y, x)$, for any $x, y \in \Omega$.

Proof: Suppose x, y are two fixed points in Ω. We define $B_\delta(y) := \{z||z - y| < \delta, z \in \Omega\}$, $\Omega_\delta := \Omega \setminus \overline{(B_\delta(x) \cup B_\delta(y))}$, $\Gamma_\delta(x) := \partial B_\delta(x)$ and $\Gamma_\delta(y) := \partial B_\delta(y)$. According to Green formula, we know that for any $u, v \in H^4(\Omega)$,

$$
\begin{aligned}
\int_{\Omega_\delta} \triangle_x^2 u \cdot v\, dx = &\int_{\partial\Omega_\delta} \frac{\partial}{\partial \nu_x} \triangle_x u \cdot v\, ds - \int_{\partial\Omega_\delta} \triangle_x u \cdot \frac{\partial}{\partial \nu_x} v\, ds \\
&+ \int_{\partial\Omega_\delta} \frac{\partial}{\partial \nu_x} u \cdot \triangle_x v\, ds \\
&- \int_{\partial\Omega_\delta} u \cdot \frac{\partial}{\partial \nu_x} \triangle_x v\, ds + \int_{\Omega_\delta} u \cdot \triangle_x^2 v\, dx
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\int_{\Omega_\delta} \triangle_z^2 G(z,x) \cdot G(z,y)\, dz = \int_{\partial\Omega_\delta} \frac{\partial}{\partial \nu_z} \triangle_z G(z,x) \cdot G(z,y)\, ds \\
&- \int_{\partial\Omega_\delta} \triangle_z G(z,x) \cdot \frac{\partial}{\partial \nu_z} G(z,y) ds + \int_{\partial\Omega_\delta} \frac{\partial}{\partial \nu_z} G(z,x) \cdot \triangle_z G(z,y) ds \\
&- \int_{\partial\Omega_\delta} G(z,x) \cdot \frac{\partial}{\partial \nu_z} \triangle_z G(z,y) ds + \int_{\Omega_\delta} G(z,x) \cdot \triangle_z^2 G(z,y) dz \\
&= I_1(\delta) - I_2(\delta) + I_3(\delta) - I_4(\delta) + \int_{\Omega_\delta} G(z,x) \cdot \triangle_z^2 G(z,y) dz
\end{aligned}
$$

Since $y, x \notin \Omega_\delta$, for any $z \in \Omega_\delta$ with $z \neq x, z \neq y$, $\triangle_z^2 G(z,y) = 0$, $\triangle_z^2 G(z,x) = 0$.

Hence we have

$$
I_1(\delta) - I_2(\delta) + I_3(\delta) - I_4(\delta) = 0.
$$

Next we will prove that $\lim_{\delta\to 0} I_2(\delta) = 0, \lim_{\delta\to 0} I_3(\delta) = 0$, $\lim_{\delta\to 0} I_1(\delta) = G(x,y)$ and $\lim_{\delta\to 0} I_4(\delta) = G(y,x)$.

Let $F(z,x) := \triangle_z G(z,x)$, then

$$
\begin{cases}
\triangle_z F(z,x) = \delta(z-x) & \text{in } \Omega \\
F(z,x)|_{\partial\Omega} = 0
\end{cases}
$$

Hence $F(z,x) \in C^\infty$ for $z \in \Omega \setminus \{x\}$ and $F(z,x) \sim \frac{1}{2\pi} \ln|z-x|\, (z \to x)$. Since $G(z,x) = G(z,y) = \triangle_z G(z,x) = \triangle_z G(z,y) = 0 \quad (z \in \partial\Omega)$

$$
I_2(\delta) = \int_{\Gamma_\delta(x)} + \int_{\Gamma_\delta(y)}.
$$

Here

$$
\int_{\Gamma_\delta(x)} \sim \frac{1}{2\pi} \int_{\Gamma_\delta(x)} \ln|z-x| \frac{\partial}{\partial \nu_z} G(z,y) dz = \frac{1}{2\pi} \int_0^{2\pi} (\delta \ln \delta) \frac{\partial}{\partial \nu} G(\delta, \theta)
$$

and $\frac{\partial}{\partial \nu_z} G(z,y)$ is bounded, so $\int_{\Gamma_\delta(x)} \to 0$ when $\delta \to 0$.

As for $\int_{\Gamma_\delta(y)}$,

$$\int_{\Gamma_\delta(y)} F(z,x)\frac{\partial}{\partial \nu_z}G(z,y)ds = F(y,x)\int_{\Gamma_\delta(y)}\frac{\partial}{\partial \nu_z}G(z,y)ds + \int_{\Gamma_\delta(y)}(F(z,x)-F(y,x))\frac{\partial}{\partial \nu_z}G(z,y)ds.$$

Here,

$$\int_{\Gamma_\delta(y)}\frac{\partial}{\partial \nu_z}G(z,y) = \int_{B_\delta(y)} F(z,y) \sim \frac{1}{2\pi}\int_0^{2\pi} d\theta \int_0^\delta r\ln r d\theta \to 0$$

Since $\triangle_z G(z,y) = F(z,y) \sim \frac{1}{2\pi}\ln|z-y| \in L^2$ near y, by the interior regularity for the Poison equation, we have $G(z,y) \in H^2$ near y and hence $\frac{\partial}{\partial \nu_z}G(z,y) \in H^{\frac{1}{2}}$ near y. So

$$\int_{\Gamma_\delta(y)}(F(z,x)-F(y,x)\frac{\partial}{\partial \nu_z}G(z,y) \to 0 \quad \text{as } \delta \to 0$$

thus we have

$$I_2(\delta) \to 0 \quad \text{as } \delta \to 0$$

By the same way , we have

$$I_3(\delta) \to 0 \quad \text{as } \delta \to 0$$

From the boundary condition for G,

$$I_1(\delta) = \int_{\partial\Omega_\delta}\frac{\partial}{\partial \nu_z}F(z,x)G(z,y)ds = (\int_{\Gamma_\delta(x)} + \int_{\Gamma_\delta(y)})\frac{\partial}{\partial \nu_z}F(z,x)G(z,y)ds.$$

Hence we know that $F(z,x) \in C^\infty$ near y, and $G(z,y) \in H^2$ which means $G(z,y) \in C^{1-\epsilon}$. So we have

$$\int_{\Gamma_\delta(y)}\frac{\partial}{\partial \nu_z}F(z,x)G(z,y)ds \to 0 \quad (\delta \to 0).$$

As for $\int_{\Gamma_\delta(x)}$, we have

$$\int_{\Gamma_\delta(x)}\frac{\partial}{\partial \nu_z}F(z,x)G(z,y) = G(x,y)\int_{\Gamma_\delta(x)}\frac{\partial}{\partial \nu_z}F(z,x) + \int_{\Gamma_\delta(x)}(G(z,y)-G(x,y))\frac{\partial}{\partial \nu_z}F(z,x).$$

Since $F(z,x) \sim \frac{1}{2\pi}\ln|z-x|$, $\frac{\partial}{\partial \nu_z}F(z,x) \sim \frac{1}{2\pi|z-x|}$, $G(z,y)-G(x,y) = O(|z-x|)$

for z near x,

$$\int_{\Gamma_\delta(x)} (G(z,y) - G(x,y)) \frac{\partial}{\partial \nu_z} F(z,x) ds \to 0 (\delta \to 0)$$

and

$$\int_{\Gamma_\delta(x)} \frac{\partial}{\partial \nu_z} F(z,x) ds \sim \frac{1}{2\pi} \int_0^{2*\pi} \delta \delta^{-1} d\theta = 1 (\delta \to 0).$$

Thus $I_1(\delta) \to G(x,y)$ as $\delta \to 0$.

By the same way, we can prove that $I_4(\delta) \to G(y,x)$ as $\delta \to 0$. This completes the proof.

7 Appendix B: Algorithm of computing $G(x,y)$ and the second order derivatives of $f_*(x)$

Assume $\Omega = (0, L) \times (0, 2\pi)$ and fix $y \in \Omega$. The problem of solving

$$\begin{cases} \triangle_x^2 G(x,y) = \delta(x-y) \quad \text{in } \Omega \\ G(x,y)|_{\partial\Omega} = \triangle_x G(x,y)|_{\partial\Omega} = 0 \end{cases}$$

can be transformed into solving

$$\begin{cases} \triangle_x F(x,y) = \delta(x-y) \quad \text{in } \Omega \\ F(x,y)|_{\partial\Omega} = 0 \end{cases}$$

and

$$\begin{cases} \triangle_x G(x,y) = F(x-y) \quad \text{in } \Omega \\ G(x,y)|_{\partial\Omega} = 0. \end{cases}$$

Define

$$u_k(x) = \sin \frac{k_1 \pi x_1}{L} \sin \frac{k_2 x_2}{2},$$

where $x = (x_1, x_2), k = (k_1, k_2)$. Then, by a direct computation, $F(x,y)$ and $G(x,y)$ are given by

$$F(x,y) = \sum_k p_k(y) u(x) = \sum_k p_k(y) \sin \frac{k_1 \pi x_1}{L} \sin \frac{k_2 x_2}{2},$$

$$G(x,y) = \sum_k q_k(y) u_k(x),$$

where

$$p_k(x) = \frac{-u_k(x)}{(\frac{k_1^2\pi^2}{L^2} + \frac{k_2^2}{2^2})\frac{\pi L}{2}}$$

$$q_k(y) = \frac{-p_k(y)}{(\frac{k_1^2\pi^2}{L^2} + \frac{k_2^2}{2^2})} = \frac{u_k(y)}{(\frac{k_1^2\pi^2}{L^2} + \frac{k_2^2}{2^2})^2\frac{\pi L}{2}}.$$

So the basis functions can be computed as following

$$\begin{aligned}
a_j(x) &= \int_\Omega G(x^j, y)G(x, y)dy \\
&= \int_\Omega \sum_k q_k(x)u_k(y) \sum_k q_k(x^j)u_k(y)dy \\
&= \sum_k q_k(x)q_k(x^j) \int_\Omega u_k^2(y)dy \\
&= \sum_k q_k(x)q_k(x^j)\frac{\pi L}{2}
\end{aligned}$$

and

$$\begin{aligned}
b(x) &= \int_{\partial\Omega} \frac{\partial}{\partial\nu}\triangle_y G(y, x)\cdot\phi(y)dy + \int_{\partial\Omega}\frac{\partial}{\partial\nu}G(y, x)\cdot\varphi(y)dy \\
&= -\sum_k q_k(x)(\frac{k_1^2\pi^2}{L^2} + \frac{k_2^2}{2^2})\int_{\partial\Omega}\frac{\partial}{\partial\nu}u_k(y)\cdot\phi(y)dy \\
&\quad + \sum_k q_k(x)\int_{\partial\Omega}\frac{\partial}{\partial\nu}u_k(y)\cdot\varphi(y)dy \\
&= I(x) + J(x).
\end{aligned}$$

Now we divide $\partial\Omega$ into four parts: $\Gamma_1 : (0, L) \times 0; \Gamma_2 : L \times (0, 2\pi); \Gamma_3 : (L, 0) \times 2\pi; \Gamma_4 : (2\pi, 0) \times 0$, and we denote the integral of I, J on each part as $I_1, I_2, I_3, I_4,\ J_1, J_2, J_3, J_4$. Then $b(x) = I_1 + I_2 + I_3 + I_4 + J_1 + J_2 + J_3 + J_4 = \sum_k(I_{1k} + I_{2k} + I_{3k} + I_{4k} + J_{1k} + J_{2k} + J_{3k} + J_{4k})$ and each $I_{\ell k}$ and $J_{\ell k}$ are given as follows:

$$\begin{cases}
I_{1k} &= -p_k(x)\frac{k_2}{2}\int_0^L \sin\frac{k_1\pi y_1}{L}\cdot\phi(y_1, 0)dy_1, \\
J_{1k} &= -q_k(x)\frac{k_2}{2}\int_0^L \sin\frac{k_1\pi y_1}{L}\cdot\varphi(y_1, 0)dy_1, \\
I_{2k} &= p_k(x)\frac{k_1\pi}{L}(-1)^{k_1}\int_0^{2\pi}\sin\frac{k_2}{2}y_2\cdot\phi(L, y_2)dy_2, \\
J_{2k} &= q_k(x)\frac{k_1\pi}{L}(-1)^{k_1}\int_0^{2\pi}\sin\frac{k_2}{2}y_2\cdot\varphi(L, y_2)dy_2, \\
I_{3k} &= p_k(x)\frac{k_2}{2}(-1)^{k_2}\int_0^L \sin\frac{k_1\pi y_1}{L}\cdot\phi(y_1, 2\pi)dy_1, \\
J_{3k} &= q_k(x)\frac{k_2}{2}(-1)^{k_2}\int_0^L \sin\frac{k_1\pi y_1}{L}\cdot\varphi(y_1, 2\pi)dy_1, \\
I_{4k} &= -p_k(x)\frac{k_1\pi}{L}\int_0^{2\pi}\sin\frac{k_2}{2}y_2\cdot\phi(0, y_2)dy_2, \\
J_{4k} &= -q_k(x)\frac{k_1\pi}{L}\int_0^{2\pi}\sin\frac{k_2}{2}y_2\cdot\varphi(0, y_2)dy_2.
\end{cases}$$

Then the algorithm to compute the second order derivatives of $f_*(x)$ is given as follwos.

We immediately have

$$\begin{cases} \frac{\partial^2 q_k}{\partial x_1 \partial x_1}(x) = -\frac{{k_1}^2\pi^2}{L^2} q_k(x), \\ \frac{\partial^2 q_k}{\partial x_1 \partial x_1}(x) = -\frac{{k_2}^2}{2^2} q_k(x), \\ \frac{\partial^2 q_k}{\partial x_1 \partial x_2}(x) = -\frac{k_1\pi}{L}\frac{k_2}{2} q_k(x). \end{cases}$$

Hence,

$$\begin{cases} \frac{\partial^2 a_j}{\partial x_1 \partial x_1}(x, x^j) = \sum_k -\frac{{k_1}^2\pi^2}{L^2} q_k(x) q_k(x^j), \\ \frac{\partial^2 a_j}{\partial x_2 \partial x_2}(x, x^j) = \sum_k -\frac{{k_2}^2}{2^2} q_k(x) q_k(x^j), \\ \frac{\partial^2 a_j}{\partial x_1 \partial x_2}(x, x^j) = \sum_k \frac{k_1\pi}{L}\frac{k_2}{2} q_k(x) q_k(x^j). \end{cases}$$

Since for

$$\phi(x) = 0, \quad \varphi(x) = 0, \quad x \in \partial\Omega$$

we have

$$\begin{cases} \frac{\partial^2 b}{\partial x_1 \partial x_1} = 0, \\ \frac{\partial^2 b}{\partial x_2 \partial x_2} = 0, \\ \frac{\partial^2 b}{\partial x_1 \partial x_2} = 0. \end{cases}$$

Therefore

$$\begin{cases} \frac{\partial^2 f_*}{\partial x_1 \partial x_1}(x) = \sum_{j=1}^N c_j \frac{\partial^2 a_j}{\partial x_1 \partial x_1} = \sum_{j=1}^N c_j \sum_k -\frac{{k_1}^2\pi^2}{L^2} q_k(x) q_k(x^j), \\ \frac{\partial^2 f_*}{\partial x_2 \partial x_2}(x) = \sum_{j=1}^N c_j \frac{\partial^2 a_j}{\partial x_2 \partial x_2} = \sum_{j=1}^N c_j \sum_k -\frac{{k_2}^2}{2^2} q_k(x) q_k(x^j), \\ \frac{\partial^2 f_*}{\partial x_1 \partial x_2}(x) = \sum_{j=1}^N c_j \frac{\partial^2 a_j}{\partial x_1 \partial x_2} = \sum_{j=1}^N c_j \sum_k \frac{k_1\pi}{L}\frac{k_2}{2} q_k(x) q_k(x^j). \end{cases}$$

APPLICATION OF NUMERICAL DIFFERENTIATION TO DETECTING AN EARLY STAGE BREAST CANCER BY MRE

G. NAKAMURA, S. WANG AND M. SUGA

Abstract. As an application of the numerical differentiation, we propose a method for an approximate identification of the early stage breast cancer by MRE.

1. Introduction

Numerical differentiation is a problem to determine the derivatives of a function from some input data on scattered points. It arises from many scientific researches and applications. The main difficulty is that, it is an ill-posed problem, which means, the small error of measurement will cause huge error in the computed derivatives ([7], [9], [14]). The Tikhonov regularization for treating the numerical differentiation problem has been shown quite effective ([5], [7], [11]). A simple but very useful solution for the one-dimensional case based on Tikhonov regularization method has been developed in ([17], [9]). For the two dimensional case, a scheme for computing the first order derivatives is given in [18] and the second order derivatives is given in [12].

A counterpart of this method is that it also gives a way to detect the discontinuity in the derivatives of a function which we want to identify. This was shown by [17] for the one-dimensional case.

In this paper, we extend this to higher dimensional case. As a very interesting application of this method, we consider the inverse problem detecting an early stage breast cancer by MRE. MRI equipped with some oscillation is called MRE which provides the elastic shear wave field inside a human body caused by the oscillation as a measured data. The importance of this method especially applied for detecting an early stage cancer is as follow. For the early stage cancer the blood vessel inside

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the cancer is not much developed. Hence, MRI which detects the distribution of the hydrogen molecule is not effective. However, MRE can measure the stiffness of the cancer which is quite different from the surrounding tissue. The paper is organized as follows: in section 2, we state and review the numerical differentiation; in section 3, we state and prove a theorem showing that the scheme of the numerical differentiation can detect the discontinuity in the derivatives of a function which we want to detect; in section 4, we applied the the results in section 2 and 3 to detect the early stage breast cancer.

Not only for detecting the early stage breast cancer, our numerical differentiation can also be effectively applied to edge detection for image analysis and has big possibility to apply for identifying the crystal growing inside a liquid solution.

In our rest of our paper we describe everything in the 2 dimensional case for simplicity, it remains true for the higher dimensional case.

2. Statements of the problems and review of the known results

Suppose that $\Omega \subset \mathbb{R}^2$ is a bounded domain with a boundary $\partial\Omega$ of piecewise C^2 class and $y = y(x)$ is a function defined in Ω. Let N be a natural number and $\{x_i\}_{i=1}^N$ be a group of points in Ω. We assume that Ω is divided into N parts $\{\Omega_i\}_{i=1}^N$, and there is only one point of $\{x_i\}_{i=1}^N$ in each part. For simplicity we also assume that Ω is a rectangle and all Ω_i are congruent rectangles. We denote d_i as the diameter of Ω_i and let $d = \max\{d_i\}$.

Suppose that we know the approximate value $\tilde{y}_i$ of $y(x)$ at point x_i, i.e.

$$|\tilde{y}_i - y(x_i)| \le \delta, \qquad i = 1, 2, \cdots, N,$$

where $\delta > 0$ is a given constant called the error level.

The first problem is to find a function $f_*(x)$ which approximates function $y(x)$ such that

$$\lim_{d\to 0, \delta\to 0} \|\nabla f_* - \nabla y\|_{L^2(\Omega)} = 0.$$

By using Tikhonov regularization method, we treat this problem as the following optimization problem:

Problem 2.1. *Define a cost functional:*

$$\Phi_1(f) = \frac{1}{N}\sum_{j=1}^{N}(f(x_j) - \tilde{y}_j)^2 + \alpha\|\triangle f\|^2_{L^2(\Omega)}, \quad f \in H_1$$

where $H_1 = \{f | f \in H^2(\Omega), f|_{\partial\Omega} = \phi_1\}$ *and* $\alpha > 0$ *is a regularization parameter. We have assumed that there is a function* $\phi_1(x) \in H^{3/2}(\partial\Omega)$ *satisfying* $\|\phi_1(x) - y(x)\|_{H^{3/2}(\partial\Omega)} \leq \delta$.

The problem is then to find $f_* \in H_1$ *such that* $\Phi_1(f_*) \leq \Phi(f)$ *for every* $f \in H_1$.

The second problem is to find a function $f_*(x)$ which approximates function $y(x)$ such that

$$\lim_{d\to 0,\delta\to 0}\|\triangle f_* - \triangle y\|_{L^2(\Omega)} = 0.$$

Again by using Tikhonov regularization method, we treat this problem as the following optimization problem:

Problem 2.2. *Define a cost functional:*

$$\Phi_2(f) = \frac{1}{N}\sum_{j=1}^{N}(f(x_j) - \tilde{y}_j)^2 + \alpha\|\triangle^2 f\|^2_{L^2(\Omega)}, \quad f \in H_2$$

where $H_2 = \{f | f \in H^4(\Omega), f|_{\partial\Omega} = \phi_1, \triangle f|_{\partial\Omega} = \phi_2\}$ *and* $\alpha > 0$ *is a regularization parameter.*

The problem is then to find $f_* \in H_2$ *such that* $\Phi_2(f_*) \leq \Phi(f)$ *for every* $f \in H_2$.

For each Problem 2.1 and Problem 2.2, admit a unique solution f_* and f_{**}. Also, there are procedures to construct f_* and f_{**}. We will give the computational scheme for f_* and f_{**}. When the regularization parameter α is taken as $\alpha = \delta^2$, the error estimate for the solutions f_* and f_{**} are given in [18] and [12], respectively. Moreover, by taking a compact set $K \subset \Omega$ and cutoff functions $\chi_k \in C_0^{2k}(\Omega)$ such that $0 \leq \chi_k \leq 1$, $\chi_k = 1$ on K and replacing $\tilde{y}_i$ by $\chi_k(x_i)\tilde{y}_i$ for Problem 2.k (k=1,2), respectively, we can take ϕ_1, $\phi_2 = 0$. This avoids the singularity which may come from $\partial\Omega$ at which $\partial\Omega$ is singular. For the rest of the paper, by using this argument, we let ϕ_1, $\phi_2 = 0$.

3. The computational scheme

We first give the computational scheme for f_*.

Let $G_*(x, x^*)$ be the harmonic Green function with fixed $x^* \in \Omega$ given as the solution to

$$\begin{cases} \triangle_x G_*(x, x^*) = \delta(x - x^*) \text{ in } \Omega \\ G_*|_{\partial\Omega} = 0, \end{cases}$$

and define $a_{*j}(x)$ by

$$a_{*j}(x) = \int_\Omega G_*(x_j, x^*) G(x^*, x) dx^* . \tag{3.1}$$

By defining

$$\mathbf{A}^* = \begin{pmatrix} \alpha N + a_{*1}(x_1) & a_{*2}(x_1) & a_{*3}(x_1) & \cdots & a_{*N}(x_1) \\ a_{*1}(x_2) & \alpha N + a_{*2}(x_2) & a_{*3}(x_2) & \cdots & a_{*N}(x_2) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{*1}(x_N) & a_{*2}(x_N) & a_{*3}(x_N) & \cdots & \alpha N + a_{*N}(x_N) \end{pmatrix}$$

we consider the linear system for $\mathbf{c}^*$

$$\mathbf{A}^* \mathbf{c}^* = \mathbf{b}^*, \tag{3.2}$$

where

$$\mathbf{c}^* = \begin{pmatrix} c_1 \\ c_2 \\ \cdots \\ c_N \end{pmatrix}, \mathbf{b}^* = \begin{pmatrix} \tilde{y}_1 \\ \tilde{y}_2 \\ \cdots \\ \tilde{y}_N \end{pmatrix}.$$

Then we can compute

$$f_*(x) = \sum_{j=1}^{N} c_j a_{*j}(x) + b(x). \tag{3.3}$$

It is shown in [18] that the solution of this linear system for $\mathbf{c}$ exists and is unique. Then, the following theorem gives how to construct f_* ([18]).

Theorem 3.1. *Suppose function* $f_* = \sum_{j=1}^{N} c_j a_{*j}(x)$ *where* $a_{*j}(x)$ *is defined in* (3.1), $\{c_j\}_{j=1}^N$ *is the solution of linear system* (3.2), *then* f_* *is the solution of Problem 2.1.*

Next, we give the computational scheme for f_{**}.

Let $G_{**}(x,x^*)$ be the bi-harmonic Green function with fixed $x^* \in \Omega$ given as the solution to

$$\begin{cases} \triangle_x^2 G_{**}(x,x^*) = \delta(x-x^*) \quad \text{in } \Omega \\ G_{**}|_{\partial\Omega} = 0, \quad \triangle_x G_{**}|_{\partial\Omega} = 0. \end{cases}$$

We can obtain $G_{**}(x,y)$ by solving

$$\begin{cases} \triangle_x F(x,y) = \delta(x-y) \quad \text{in } \Omega \\ F(x,y)|_{\partial\Omega} = 0 \end{cases}$$

and

$$\begin{cases} \triangle_x G_{**}(x,y) = F(x,y) \quad \text{in } \Omega \\ G_{**}(x,y)|_{\partial\Omega} = 0\,, \end{cases}$$

and define $a_{**j}(x)$ by

$$a_{**j}(x) = \int_\Omega G_{**}(x_j,x^*)G_{**}(x^*,x)dx^*\,. \tag{3.4}$$

By defining

$$\mathbf{A}^{**} = \begin{pmatrix} \alpha N + a_{**1}(x_1) & a_{**2}(x_1) & a_{**3}(x_1) & \cdots & a_{**N}(x_1) \\ a_{**1}(x_2) & \alpha N + a_{**2}(x_2) & a_{**3}(x_2) & \cdots & a_{**N}(x_2) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{**1}(x_N) & a_{**2}(x_N) & a_{**3}(x_N) & \cdots & \alpha N + a_{**N}(x_N) \end{pmatrix}$$

we consider the linear system for $\mathbf{c}^*$

$$\mathbf{A}^{**}\mathbf{c}^{**} = \mathbf{b}^{**}, \tag{3.5}$$

where

$$\mathbf{c}^{**} = \begin{pmatrix} c_1 \\ c_2 \\ \cdots \\ c_N \end{pmatrix}, \mathbf{b}^{**} = \begin{pmatrix} \tilde{y}_1 \\ \tilde{y}_2 \\ \cdots \\ \tilde{y}_N \end{pmatrix}.$$

It is shown in [12] that the solution of this linear system for $\mathbf{c}$ exists and is unique. Then, the following theorem gives how to construct f_{**} ([12]).

Theorem 3.2. *Suppose function* $f_{**} = \sum_{j=1}^{N} c_j a_{**j}(x)$ *where* $a_{**j}(x)$ *is defined in* (3.4), $\{c_j\}_{j=1}^{N}$ *is the solution of linear system* (3.5), *then* f_{**} *is the solution of Problem 2.2.*

4. Detection of the discontinuity in the derivative

In this section, we consider Problem 2.1 again. For simplicity, we suppress the subscript 1, for instance $\Phi(f) = \Phi_1(f)$.

Let $\Omega' \subset \Omega$ be a domain sharing the same partition $\{\Omega_i\}_{i=1}^{N}$ of Ω. That is $\Omega' = \cup_{\ell=1}^{L} \Omega_{i_\ell}$. For simplicity, we write $\Omega'_\ell = \Omega_{i_\ell}$. We also denote the area of Ω by $|\Omega|$.

Theorem 4.1. *Suppose* f_* *is the minimizer of Problem 2.1. Let* $\alpha = \delta^2$. *If* $y \in C_0^0(\overline{\Omega}) \setminus H^2(\Omega')$, *then we have*

$$\|\triangle f_*\|_{L^2(\Omega')} \to \infty \quad \text{as} \quad \delta \to 0,\, N \to \infty\,. \tag{4.1}$$

Proof. We will basically follow the proof for the 1 dimensional case given in [15]. Assume that (4.1) is false. Then, there exist $C > 0$ and

$$\delta^{(m)} \to 0,\, N^{(m)} \to \infty \ (m \to \infty)$$

such that

$$\|\triangle f_*(\cdot;\delta^{(k)}, N^{(m)})\|_{L^2(\Omega')} \le C \quad (k, m = 1, 2, \cdots)\,. \tag{4.2}$$

Since $H_0^2(\Omega)$ is dense in $C_0^0(\overline{\Omega})$, there exist $y_k \in H_0^2(\Omega)\,(k = 1, 2, \cdots)$ such that

$$y_k|_{\partial\Omega} = 0,\ \|y_k - y\|_{C^0(\overline{\Omega})} \le k^{-1}\ (k = 1, 2, \cdots)$$

and

$$\sup_k \|y_k\|_{L^2(\Omega)} < \infty\,.$$

Without loss of generality, we can assume that $\delta^{(k)}\|\triangle y_k\|^2_{L^2(\Omega)} < 1$ $(k = 1, 2, \cdots)$.

Now take $k = N^{(m)}$ $(m = 1, 2, \cdots)$ and abbreviate $\delta^{(k)}$ and y_k with $k = N^{(m)}$ by $\delta^{(m)}$ and y_m, respectively. For simplicity, we will still use the same notations Ω_i, Ω'_ℓ even for the case the number of partition is $N^{(m)}$. Then, by $\Phi(f_*(\cdot;\delta^{(k)}, N^{(m)})) \le$

$\Phi(y_m)$,

$$\begin{aligned}(4.3)\qquad &\tfrac{1}{N^{(m)}}\textstyle\sum_{j=1}^{N^{(m)}}(f(x_j;\delta^{(k)},N^{(m)})-y_j^{\delta^{(k)}})^2\le\Phi(f_*(\cdot;\delta^{(k)},N^{(m)})\le\Phi(y_m)\\ &\le\tfrac{|\Omega|}{N^{(m)}}(y_m(x_j)-y_j^{\delta^{(k)}})^2+(\delta^{(k)})^2\|\triangle y_m\|^2_{L^2(\Omega)}\\ &\le 2((\delta^{(k)})^2+\tfrac{1}{(N^{(m)})^2})+\delta^{(k)}\,.\end{aligned}$$

Hence, with the easy estimate

$$\begin{aligned}(4.4)\qquad \tfrac{1}{N^{(m)}}\textstyle\sum_{\ell=1}^{L}(f_*(x_\ell;\delta^{(k)},N^{(m)})-y_\ell^{\delta^{(m)}})^2\quad &\le\tfrac{1}{N^{(m)}}\textstyle\sum_{j=1}^{N^{(m)}}(f_*(x_j;\delta^{(k)},N^{(m)}-y_j^{\delta^{(k)}})^2\\ &\le 2((\delta^{(k)})^2+\tfrac{1}{N^{(m)})^2})+\delta^{(k)}\,,\end{aligned}$$

we have

$$(4.5)\qquad \frac{1}{N^{(m)}}\sum_{\ell=1}^{L}f_*^2(x_\ell;\delta^{(k)},N^{(m)})\le 2\frac{1}{N^{(m)}}\sum_{\ell=1}^{L}(f_*(x_\ell;\delta^{(k)},N^{(m)})-y_\ell^{\delta^{(k)}})^2+2\frac{1}{N^{(m)}}\sum_{\ell=1}^{L}(y_\ell^{\delta^{(k)}})^2\,.$$

By

$$(4.6)\qquad (y_\ell^{\delta^{(k)}})^2\le 2(y_\ell^{\delta^{(k)}}-y(x_\ell))^2+2y(x_\ell)^2\le 2((\delta^{(k)})^2+\|y\|^2_{C^0(\overline{\Omega})})\,.$$

Hence, by (4.4) to (4.6),

$$(4.7)\qquad \frac{1}{N^{(m)}}\sum_{\ell=1}^{L}f_*^2(x_\ell;\delta^{(k)},N^{(m)})\le 4((\delta^{(k)})^2+\frac{1}{(N^{(m)})^2})+2\delta^{(m)}+4((\delta^{(k)})^2+\|y\|^2_{C^0(\overline{\Omega})})=:A<\infty\,.$$

Now by the embedding $H^2(\Omega)\subset C^0(\overline{\Omega})$ which is true up to 3 dimension and the definition of the Riemann integral, for any $k\,(k=1,2,\cdots)$, there exists a positive integer $M(k)$ such that

$$\int_{\Omega'}|f_*^2(x;\delta^{(k)},N^{(m)})|^2\,dx\le 2A$$

if k, m are large enough. Hence, there exists a monotonically decreasing sequence of positive integers $m_k\,(k=1,2,\cdots)$ such that

$$\int_{\Omega'}|f_*^2(x;\delta^{(k)},N^{(m_k)})|^2\,dx\le 2A\,(k=1,2,\cdots).$$

Combining this with (4.2), $\{\|f_*(\cdot;\delta^{(k)},N^{(m_k)})\|_{H^2(\Omega7)}\}_{k-1}^{\infty}$ is bounded. By the weak compactness of $H^2(\Omega')$, the compactness of the embedding $H^2(\Omega')\subset C^0(\overline{\Omega'})$ and taking a subsequence of $\{f_*(\cdot;\delta^{(k)},N^{(m_k)})\}_{j=1}^{\infty}$ if necessary, there exists $\tilde f\in H^2(\Omega')$ such that

$$(4.8)\qquad \|f_*(\cdot;\delta^{(k)},N^{(m_k)})-\tilde f\|_{C^0(\overline{\Omega'})}\to 0\quad (k\to\infty)\,.$$

Now, let $0 < \varepsilon < 1$. By the uniform continuity of $y - \tilde{f}$ on $\overline{\Omega'}$ and (4.8), there exist $K \in \mathbb{N}$ such that for any $k \geq K$, we have

$$|(y(x) - \tilde{f}(x)) - (y(x_\ell) - \tilde{f}(x_\ell))| < \sqrt{\frac{\varepsilon}{2|\Omega|}} \; (x \in \Omega'_\ell,\, 1 \leq \ell L). \tag{4.9}$$

By using (4.9) first and then (4.5), we have

(4.10)

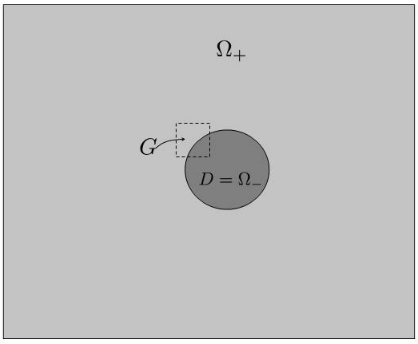
$$\begin{aligned}\|y - \tilde{f}\|^2_{L^2(\Omega')} &= \textstyle\sum_{\ell=1}^L \int_{\Omega'_\ell} (y(x) - \tilde{f}(x))^2\, dx \\ &\leq 2\tfrac{|\Omega|}{N^{(m_k)}} \textstyle\sum_{\ell=1}^L (y(x_\ell) - \tilde{f}(x_\ell))^2 + \varepsilon \\ &\leq 6\tfrac{|\Omega|}{N^{(m_k)}} \textstyle\sum_{\ell=1}^L \{(y(x_\ell) - y_\ell^{\delta^{(m_k}})^2 + (y_\ell^{\delta^{(m_k)}} - f_*(x_\ell; \delta^{(k)}, N^{(m_k)}))^2 \\ &\qquad + (f_*(x_\ell; \delta^{(k)}, N^{(m_k)}) - \tilde{f}(x_\ell))^2)\} + \varepsilon \\ &\leq 6|\Omega|(\delta^{(k)})^2 + 12|\Omega|((\delta^{(k)})^2 + \tfrac{1}{(N^{(m_k)})^2}) + 6|\Omega|\delta^{(k)} + 6\varepsilon^2 + \varepsilon\,.\end{aligned}$$

Hence, letting $k \to \infty$, we have

$$\|y - \tilde{f}\|^2_{L^2(\Omega')} < 7\varepsilon\,.$$

Therefore, $y = \tilde{f} \in H^2(\Omega')$. This is a contradiction.

5. Application to detecting the early stage breath cancer



We propose a scheme for detecting an early stage breast cancer as an application of Theorem 4.1. Note that we are assuming that there is a discontinuity in their elastic properties between the cancer and the surrounding tissue. Of course this is an artificial assumption. But to analyze the measured data obtained by Suga's experiment, we need to assume this. If we can succeed analyzing the experimental data effectively, we will consider the case that there is not any discontinuity. Again for simplicity we describe the scheme for the 2 dimensional case.

Let Ω be rectangle given in Section 2. We consider Ω as an elastic medium with a compactly embedded inclusion D with C^2 boundary ∂D. D corresponds to a cancer in some part Ω of the two dimensional cross section of a breath. For convenience we denote D by Ω_+ and its exterior in Ω by Ω_-. We consider that the stiffness (λ_+, μ_+) of Ω_+ is much larger than the stiffness (λ_-, μ_-) of Ω_- and has a clear jump across $\partial\Omega_+$. Since we are interested in the early stage brest cancer which is small, we can assume that the stiffness in Ω_+ is uniform. That is λ_+, μ_+ are constant.

In the MRE measurement, the elastic wave field u is incompressible. That is $\text{div}E(u) = 0$ for the strain tensor $E(u) := 1/2(\nabla u + {}^t(\nabla u))$. So, we will neglect $\text{div}E(u)$. Then, it is easy to see that in Ω_+, the wave field u_+ in Ω_+ satisfies

$$\mu_+ \triangle u_+ + k^2 u_+ = 0 \quad \text{in} \quad \Omega_+ \,, \tag{5.1}$$

where $k > 0$ is the wave number. Due to the discontinuity of the stiffness across Ω_+, the numerically simulated wave field u is continuous everywhere, but its gradient ∇u has a clear discontinuity at $\partial\Omega_+$ which means that it is not in H^2 in any neighborhood of each point of $\partial\Omega_+$.

Basing on these facts, our scheme for detecting $\partial\Omega_+$ and the shear stiffness μ_+ is as follows.

Step 1: Use Theorem 4.1 to detect $\partial\Omega_+$.

Step 2: Use the result for Problem 2.2 to compute $\triangle u_+$ and then use (5.1) to obtain μ_+.

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Recovery of the Elasticity and Viscosity from the Interior Measurements

Gen Nakamura*

1 Introduction

In the method of dynamic MR-Elastography, it is reasonable to consider not only the elastic properties of the material but also the viscous properties of the material. There are various models to introduce viscosity into the elastic equation. The simplest model is the so-called Voigt model.

That is, for any time $t > 0$, and a point $x = (x_1, \cdots, x_n)$ in a bounded domain $\Omega \subset \mathbb{R}^n (n = 2, 3)$ whose boundary $\partial\Omega$ is $\mathcal{C}^\infty$ smooth, the displacement $u(x,t)$ satisfies the equation:

$$\rho(x)\partial_t^2 u_i - \sum_k \frac{\partial}{\partial_k} \sum_{lm} \lambda_{ilkm}(x) u_{lm} - \sum_k \frac{\partial}{\partial_k} \sum_{lm} \eta_{ilkm}(x) \partial_t u_{lm} = 0, \quad (1.1)$$

where

$$u_{ij} := \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right),$$

$\rho(x) > 0$ is the density, the elasticity tensor λ_{ilkm} and the viscosity tensor η_{ilkm} satisfy the symmetries:

$$\lambda_{ilkm} = \lambda_{kilm} = \lambda_{ikml} = \lambda_{lmik},$$

$$\eta_{ilkm} = \eta_{kilm} = \eta_{ikml} = \eta_{lmik}.$$

If we assume that the material is isotropic and incompressible, the Voigt model (1.1) reduces to a scalar equation with shear modulus $a(x)$ and viscosity coefficient $b(x)$.

For simplicity, we assume $n = 2$ and $a(x), b(x), \rho(x) \in \mathcal{C}^\infty(\bar{\Omega})$ satisfy the following condition

$$a(x), b(x), \rho(x) > 0 \quad \text{on} \quad \bar{\Omega}.$$

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Then, the forward problem is as follow.

Forward Problem:

For any $f \in \mathcal{C}^2([0,\infty); H^{\frac{3}{2}}(\partial\Omega))$, to find a solution $u \in \mathcal{C}^0([0,\infty); H^1(\Omega)) \cap \mathcal{C}^1([0,\infty); L^2(\Omega))$ to

$$\begin{cases} \rho(x)\partial_t^2 u - \nabla\cdot(a(x)\nabla u + b(x)\nabla u_t) = 0 & \text{in} \quad (0,\infty)\times\Omega, \\ u = f & \text{on} \quad (0,\infty)\times\partial\Omega, \\ u = u_t = 0 & \text{on} \quad \{0\}\times\Omega. \end{cases} \tag{1.2}$$

It is well known the forward problem is well posed. We denote the solution u to (1.2) by $u = u(f)$. Moreover, we have the following:

Proposition 1.1. *For any $u_0 \in H^1(\Omega)$, $u_1 \in L^2(\Omega)$, $F \in \mathcal{C}^0([0,\infty); L^2(\Omega))$, there exist a unique solution $u \in \mathcal{C}^0([0,\infty); H^1(\Omega)) \cap \mathcal{C}^1([0,\infty); L^2(\Omega))$ to*

$$\begin{cases} \rho(x)\partial_t^2 u - \nabla\cdot(a(x)\nabla u + b(x)\nabla u_t) = F & \textit{in} \quad (0,\infty)\times\Omega, \\ u = 0 & \textit{on} \quad (0,\infty)\times\partial\Omega, \\ u = u_0, \quad u_t = u_1 & \textit{on} \quad \{0\}\times\Omega. \end{cases} \tag{1.3}$$

Also, there exist a constant $c_0 > 0$ independent of u_0, u_1, F such that

$$\|u(t)\|_{H^1(\Omega)} + \|\partial_t u(t)\|_{L^2(\Omega)} = O(e^{-c_0 t}) \quad (t \to \infty). \tag{1.4}$$

Based on the well posedness of the forward problem, we formulate the own inverse problem as follow.

Inverse Problem:

Suppose $a(x), b(x), \rho(x)$ are unknown. Reconstruct $a(x)$, $b(x)$, $\rho(x)$ from $u(f)$ in $(0,T)\times\bar{\Omega}$ for finitely many f's, where $u = u(f)$ is the solution of (1.2).

Theorem 1.2. *There is a reconstruction procedure for this inverse problem.*

The details of the reconstruction procedure will be given later.

2 The dominant part of $u(f)$

Lemma 2.1. *Let $0 < \omega < c_0$ and $f(x.t) = e^{-\omega t} g(x)$ with $g(x) \in H^{\frac{1}{2}}(\partial\Omega)$. Then, (1.2) has a unique solution $u \in \mathcal{C}^0([0,\infty); H^1(\Omega)) \cap \mathcal{C}^1([0,\infty); L^2(\Omega))$ with dominant part $e^{-\omega t} v(x)$ where $v(x)$ solves*

$$\begin{cases} \nabla\cdot(a\nabla v - \omega b\nabla v) - \rho\omega^2 v = 0 & \textit{in} \quad \Omega, \\ v = g & \textit{on} \quad \partial\Omega, \end{cases} \tag{2.1}$$

Hence, we can say that we know v in Ω if we know $u(e^{-\omega t}g)$ in Ω. From now on we only consider ω small enough so that

$$\gamma(\omega) := a - \omega b > 0 \quad \text{on} \quad \bar{\Omega}. \tag{2.2}$$

3 Application of oscillating-decaying solutions

Lemma 3.1. *By using finitely many oscillating-decaying solutions (abbreviated by OD solutions), and two different ω's, we can approximately recover $a(x), b(x), \rho(x)$ and their derivatives on $\partial\Omega$.*

Hence we can approximately extend $a(x), b(x), \rho(x)$ smoothly outside Ω as positive functions.

Let's take a box $\widetilde{\Omega} \supset \bar{\Omega}$. We round the corners of $\widetilde{\Omega}$ so that $\partial\widetilde{\Omega}$ is smooth. We still use the same notation $\widetilde{\Omega}$ (see Figure. 1) to denote the domain we obtain by this extension and assume that $\widetilde{\Omega} \supset \bar{\Omega}$.

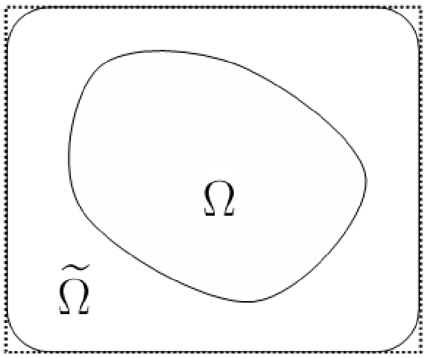

Figure. 1: Diagram of the domains

4 Recover $a(x), b(x)$ and $\rho(x)$

For simplicity, we will work on $\widetilde{\Omega}$ instead of Ω to illustrate our reconstruction procedure.

Lemma 4.1. *Consider the solution v to* (2.1) *for $\omega = \omega_1, \omega_2, \omega_3$ and denote the associated v's by v_1, v_2, v_3. By an algebraic manipulation, we can delete the terms with ρ and obtain a first order equation with leading term:*

$$\left\{ \begin{matrix} \omega_2^2 v_2 \nabla v_1 \cdot \nabla(a - \omega_1 b) - \omega_1^2 v_1 \nabla v_2 \cdot \nabla(a - \omega_2 b) \\ \omega_3^2 v_3 \nabla v_1 \cdot \nabla(a - \omega_1 b) - \omega_1^2 v_1 \nabla v_3 \cdot \nabla(a - \omega_3 b) \end{matrix} \right\}. \tag{4.1}$$

Lemma 4.2. *By using an OD solution we can continue $a(x), b(x)$ from one side of $\widetilde{\Omega}$ by solving the Cauchy problem for a hyperbolic system with respect to $a(x), b(x)$. Here the initial surface is the side of $\widetilde{\Omega}$ mentioned above.*

Remark 4.3. *Of course, once we recover $a(x), b(x)$ in a subdomain of $\widetilde{\Omega}$, we can recover $\rho(x)$ in the subdomain if $g \not\equiv 0$ on $\partial\widetilde{\Omega}$. This is because the set of points at which v does not vanish is dense in $\widetilde{\Omega}$ due to the unique continuation property if the partial differential equation given in* (2.1).

Now we want to repeat this reconstruction procedure to recover $a(x), b(x)$ and $\rho(x)$.

For that, we consider some domain $\widetilde{\Omega}_1$, such that $\bar{\widetilde{\Omega}}_1 \subset \widetilde{\Omega}$, $a(x), b(x), \rho(x)$ are known in $\widetilde{\Omega} \backslash \bar{\widetilde{\Omega}}_1$ and its shape is analogous to that of $\widetilde{\Omega}$. We note that $\widetilde{\Omega}_1$ does not have to contain Ω. We again use OD solution to set up the Cauchy problem for a hyperbolic system with respect to $a(x), b(x)$ near one side of $\widetilde{\Omega}_1$ which is parallel to the previous side of $\widetilde{\Omega}_1$. This OD solution needs the full information about $a(x), b(x), \rho(x)$ in $\widetilde{\Omega}_1$. However, the dominant part of its Cauchy data can be obtained from $(a, b)|_{\partial\widetilde{\Omega}_1}$.

Now, by solving the Cauchy problem for the partial differential equation given in (2.1) with an incomplete Cauchy data (i.e. the previous dominant part of the Cauchy data), we can generate a Dirichlet data g in (2.1) such that the associated solution v to (2.1) approximates the Cauchy data of the OD solution on the previous side of $\widetilde{\Omega}_1$. For solving the Cauchy problem approximately with incomplete Cauchy data, we use the alternating method.

Since the solvability of the Cauchy problem of hyperbolic system only depends on its coefficients and the coefficients are generated by the OD solution, we can completely control the solvability if we assume some a priori bounds on $a(x), b(x)$.

Problems on Modeling, Mathematical and Numerical Analysis of Visco-Elasticity

藤原宏志

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式番号 (Y**) は，山本先生の資料の式番号です.

1 山本先生の理論

領域 $\Omega \subset R^3$ を占める Voigt 型等方粘弾性体の運動方程式は次のとおり.

$$\rho\frac{\partial^2 u_i}{\partial t^2} - \sum_j \frac{\partial}{\partial x_j}\left(\mu\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)\right) - \frac{\partial}{\partial x_i}\left(\lambda\nabla\cdot u\right) - \sum_j \frac{\partial}{\partial x_j}\left(\eta\left(\frac{\partial \dot{u}_i}{\partial x_j}+\frac{\partial \dot{u}_j}{\partial x_i}\right)\right) - \frac{\partial}{\partial x_i}\left(\tilde{\zeta}\nabla\cdot\dot{u}\right) = F_i, \quad i=1,2,3. \tag{Y20'}$$

ここで,

$u=(u_1,u_2,u_3)$, $u_i=u_i(t,x)$ は静止状態からの x_i 方向の変位, $\dot{u}=\partial u/\partial t$,
$\rho=\rho(x)$ は密度, $F_i=F_i(t,x)$ は体積力,
$\mu=\mu(x)$ は剛性率, $\lambda=\lambda(x)$ はもう一方の Lamé 係数,
$\eta=\eta(x)$ は粘性, $\tilde{\zeta}=\tilde{\zeta}(x)$ は実数値で，$\zeta=\tilde{\zeta}+(2/3)\eta$ が体積粘性率,
$\nabla=\left(\frac{\partial}{\partial x_1},\frac{\partial}{\partial x_2},\frac{\partial}{\partial x_3}\right)$

を表す．以下 ρ は Ω 上で定数とし，$F_i=0$ とする.

力に対する変形が，位置に依存して位相 $\theta=\theta(x)$ だけ遅れて生じることを考える．すなわち，

$$u(t,x)=A(x)\cos(\omega t-\theta(x))$$

なる運動を考えると，

$$\begin{aligned} u(t,x) &= A(x)\big(\cos\omega t\ \cos\theta+\sin\omega t\ \sin\theta\big) \\ &= (A\cos\theta)\ \cos\omega t+(A\sin\theta)\ \sin\omega t. \end{aligned}$$

これより，次の $u(t,x)$ で表される運動を考える．

$$u(t,x) = \phi(x)\cos(\omega t) + \psi(x)\sin(\omega t) \tag{Y22'}$$

すなわち (Y20') の解として (Y22') の形のものを考える．これを time-harmonic solution と云うことにする．ω は考える周波数，$\phi = (\phi_1, \phi_2, \phi_3), \psi = (\psi_1, \psi_2, \psi_3)$ である．

time-harmonic solution を式 (Y20') に代入すると，ϕ, ψ について次を得る ($i = 1, 2, 3$)．

$$\begin{aligned}
&-\rho\omega^2\phi_i - \sum_j \frac{\partial}{\partial x_j}\left(\mu\left(\frac{\partial \phi_i}{\partial x_j} + \frac{\partial \phi_j}{\partial x_i}\right)\right) - \frac{\partial}{\partial x_i}(\lambda\nabla\cdot\phi) \\
&\qquad - \omega\sum_j \frac{\partial}{\partial x_j}\left(\eta\left(\frac{\partial \psi_i}{\partial x_j} + \frac{\partial \psi_j}{\partial x_i}\right)\right) - \omega\frac{\partial}{\partial x_i}\left(\tilde{\zeta}\nabla\cdot\psi\right) = 0, \qquad \text{(Y31a')} \\
&-\rho\omega^2\psi_i - \sum_j \frac{\partial}{\partial x_j}\left(\mu\left(\frac{\partial \psi_i}{\partial x_j} + \frac{\partial \psi_j}{\partial x_i}\right)\right) - \frac{\partial}{\partial x_i}(\lambda\nabla\cdot\psi) \\
&\qquad + \omega\sum_j \frac{\partial}{\partial x_j}\left(\eta\left(\frac{\partial \phi_i}{\partial x_j} + \frac{\partial \phi_j}{\partial x_i}\right)\right) + \omega\frac{\partial}{\partial x_i}\left(\tilde{\zeta}\nabla\cdot\phi\right) = 0. \qquad \text{(Y31b)}
\end{aligned}$$

非圧縮性の仮定 $\nabla\cdot u = \nabla\cdot\dot{u} = 0$, すなわち

$$\nabla\cdot\phi = \nabla\cdot\psi = 0 \tag{Y34}$$

および振動方向についての仮定

$$\sum_j \frac{\partial \mu}{\partial x_j}\frac{\partial \phi_j}{\partial x_i} = 0, \quad \sum_j \frac{\partial \mu}{\partial x_j}\frac{\partial \psi_j}{\partial x_i} = 0, \quad \sum_j \frac{\partial \eta}{\partial x_j}\frac{\partial \phi_j}{\partial x_i} = 0, \quad \sum_j \frac{\partial \eta}{\partial x_j}\frac{\partial \psi_j}{\partial x_i} = 0 \tag{Y43}$$

をすると，(Y31) は次で近似される．

$$\begin{aligned}
&-\rho\omega^2\phi_i - \nabla\cdot(\mu\nabla\phi_i) - \omega\nabla\cdot(\eta\nabla\psi_i) = 0, \quad \text{in } \Omega, \qquad \text{(Y44a')} \\
&-\rho\omega^2\psi_i - \nabla\cdot(\mu\nabla\psi_i) + \omega\nabla\cdot(\eta\nabla\phi_i) = 0, \quad \text{in } \Omega. \qquad \text{(Y44b)}
\end{aligned}$$

さらに均質性 (μ, η が定数; 仮定 (Y45)) を仮定すると，次を得る．

$$\begin{aligned}
&-\rho\omega^2\phi_i - \mu\triangle\phi_i - \omega\eta\triangle\psi_i = 0, \quad \text{in } \Omega, \qquad \text{(Y46a)} \\
&-\rho\omega^2\psi_i - \mu\triangle\psi_i + \omega\eta\triangle\phi_i = 0, \quad \text{in } \Omega. \qquad \text{(Y46b)}
\end{aligned}$$

仮定 (Y43) の意味はつくか? 均質性の仮定

$$\frac{\partial \mu}{\partial x_i} = 0, \quad \frac{\partial \eta}{\partial x_i} = 0 \tag{Y45}$$

をすると，仮定 (Y43) は満たされる．

2　順問題の設定

以下，$x_1(i=1)$ の方向を考えるものとして添字は省略する．順問題では ϕ, ψ が未知である．$i=1$ の方向のみを考えて，$\phi=\phi(x), \psi=\psi(x)$ はスカラ函数としていることに注意する．

運動方程式として (Y44) から出発する．すなわち，

$$-\rho\omega^2\phi - \nabla\cdot(\mu\nabla\phi) - \omega\nabla\cdot(\eta\nabla\psi) = 0, \quad \text{in } \Omega, \tag{1a}$$

$$-\rho\omega^2\psi - \nabla\cdot(\mu\nabla\psi) + \omega\nabla\cdot(\eta\nabla\phi) = 0, \quad \text{in } \Omega. \tag{1b}$$

式 (1) に境界条件を加え，順問題の設定をおこなう．順問題の境界条件とは，物体表面での変位や応力を与えるのが一般的である．このとき (1) に境界条件を合わせた系 (問題設定) が適切 (well-posed) になるように与えることが望ましい．系が適切であるとは，系の (i) 解が存在して，(ii) 解はただひとつであり，(iii) 境界値や係数に対して解が連続である，の 3 条件が満たされることである．

2.1　境界条件の設定 (表面の変位)

物体の表面 $\partial\Omega$ の一部 Γ_d で静止状態からの変位 (振幅) U が与えられているとする．このとき ϕ, ψ の Dirichlet 値を次で設定する．

$$\phi = U, \quad \psi = 0.$$

2.2　境界条件の設定 (表面応力)

Γ_d を除いた境界 $\Gamma = \partial\Omega \setminus \Gamma_d$ で，表面応力 (体積力とは異なる) が与えられているとする．この表面応力を表す境界条件を，(1) の弱形式から考察する．式 (1) に任意のテスト函数 $v=v(x)$ を乗じて Ω 上で積分すると

$$-\rho\omega^2\int_\Omega \phi v\,dx - \int_\Omega \nabla\cdot(\mu\nabla\phi)v\,dx - \omega\int_\Omega \nabla\cdot(\eta\nabla\psi)v\,dx = 0, \tag{2a}$$

$$-\rho\omega^2\int_\Omega \psi v\,dx - \int_\Omega \nabla\cdot(\mu\nabla\psi)v\,dx + \omega\int_\Omega \nabla\cdot(\eta\nabla\phi)v\,dx = 0. \tag{2b}$$

Green の公式により

$$-\rho\omega^2\int_\Omega \phi v\,dx - \left\{\int_{\partial\Omega} \mu\frac{\partial\phi}{\partial n}v\,d\sigma - \int_\Omega \mu\nabla\phi\cdot\nabla v\,dx\right\} - \omega\left\{\int_{\partial\Omega} \eta\frac{\partial\psi}{\partial n}v\,d\sigma - \int_\Omega \eta\nabla\psi\cdot\nabla v\,dx\right\} = 0, \tag{3a}$$

$$-\rho\omega^2\int_\Omega \psi v\,dx - \left\{\int_{\partial\Omega} \mu\frac{\partial\psi}{\partial n}v\,d\sigma - \int_\Omega \mu\nabla\psi\cdot\nabla v\,dx\right\} + \omega\left\{\int_{\partial\Omega} \eta\frac{\partial\phi}{\partial n}v\,d\sigma - \int_\Omega \eta\nabla\phi\cdot\nabla v\,dx\right\} = 0. \tag{3b}$$

ここで $n=(n_1,n_2,n_3), n_i=n_i(x)$ は $\partial\Omega$ の外向き単位法線である．

弱形式 (3) から，$\partial\Omega$ における (1) の flux は次のとおり．

$$\mu\frac{\partial\phi}{\partial n} + \omega\eta\frac{\partial\psi}{\partial n} = f_1, \quad \mu\frac{\partial\psi}{\partial n} - \omega\eta\frac{\partial\phi}{\partial n} = f_2. \tag{4}$$

この f_1, f_2 が Γ における表面応力であると考えられる．(表面応力であると考えたい)

以上をまとめて，粘弾性の定常状態の運動として，次の順問題を設定する．

$$-\rho\omega^2\phi - \nabla\cdot(\mu\nabla\phi) - \omega\nabla\cdot(\eta\nabla\psi) = 0, \quad \text{in } \Omega, \tag{5a}$$

$$-\rho\omega^2\psi - \nabla\cdot(\mu\nabla\psi) + \omega\nabla\cdot(\eta\nabla\phi) = 0, \quad \text{in } \Omega, \tag{5b}$$

$$\phi = U, \quad \psi = 0, \quad \text{on } \Gamma_d, \tag{5c}$$

$$\mu\frac{\partial\phi}{\partial n} + \omega\eta\frac{\partial\psi}{\partial n} = f_1, \quad \mu\frac{\partial\psi}{\partial n} - \omega\eta\frac{\partial\phi}{\partial n} = f_2, \quad \text{on } \Gamma. \tag{5d}$$

μ, η に然るべき条件を課すと，この問題設定は適切か?

菅先生の実験を念頭において，力を加えない (stress free) 条件を考える．これは $f_1 = f_2 = 0$ で表される．(このとき (5d) は $\partial\phi/\partial n = \partial\psi/\partial n = 0$ on Γ と同値である．)

2.3 順問題の数値計算

問題 (5) に対応する弱形式は次のとおり: Find $\phi(x), \psi(x)$ such that

$$\phi = U, \quad \psi = 0 \quad \text{on } \Gamma_d, \tag{6a}$$

$$-\rho\omega^2\int_\Omega \phi v\,dx + \int_\Omega \mu\nabla\phi\cdot\nabla v\,dx + \omega\int_\Omega \eta\nabla\psi\cdot\nabla v\,dx = \int_\Gamma f_1 v\,d\sigma, \tag{6b}$$

$$-\rho\omega^2\int_\Omega \psi v\,dx + \int_\Omega \mu\nabla\psi\cdot\nabla v\,dx - \omega\int_\Omega \eta\nabla\phi\cdot\nabla v\,dx = \int_\Gamma f_2 v\,d\sigma, \tag{6c}$$

for any v with $v|_{\partial\Omega\setminus\Gamma} = 0$.

有限要素法で (6) の数値計算をおこない，ϕ, ψ の近似解を構成する．これをもって (5) の近似解とする．

然るべき条件のもとで有限要素法の収束性と安定性を示せ．(ϕ, ψ, v は H^1, μ, η は区分的に C^∞?)

3 係数決定の逆問題

式 (1) (あるいは系 (5)) を満たす ϕ, ψ がわかっているとする．Ω に含まれるある領域 Ω' において μ, η が定数である と仮定すると，(1) は (46) 式になる．すなわち次を満たす．

$$-\rho\omega^2\phi - \mu\triangle\phi - \omega\eta\triangle\psi = 0, \quad \text{in } \Omega', \tag{7a}$$

$$-\rho\omega^2\psi - \mu\triangle\psi + \omega\eta\triangle\phi = 0, \quad \text{in } \Omega'. \tag{7b}$$

μ, η について代数的に解くと次を得る．

$$\mu = -\rho\omega^2\frac{\phi\triangle\phi + \psi\triangle\psi}{(\triangle\phi)^2 + (\triangle\psi)^2}, \tag{8a}$$

$$\eta = -\rho\omega\frac{\phi\triangle\psi - \psi\triangle\phi}{(\triangle\phi)^2 + (\triangle\psi)^2}. \tag{8b}$$

μ, η がともに定数であることがわかっていれば，剛性率 μ と粘性率 η が求められる．

$(\triangle\phi)^2 + (\triangle\psi)^2 = 0$ なら (7) より $\phi = \psi = 0$，すなわち，この点は停留点である．そのような点が存在しないことを証明できるか?

3.1 係数同定の数値計算

空間方向を 2 次元で考える．領域 Ω に間隔 $h(>0)$ の格子点を考える．すなわち，$x_i = ih, y_j = jh$ として格子点 $\{(x_i, y_j) \in \Omega\,;\, i, j \in \mathbb{Z}\}$ を考える．ϕ, ψ の格子点上での値がわかっているとする．

式 (8) で必要な $\triangle\phi$ の計算には次の近似を利用する．$\phi \in C^4$ と仮定して Taylor 展開より

$$\phi(x_i \pm h, y_j) = \phi(x_i, y_j) \pm \frac{\partial\phi}{\partial x}(x_i, y_j)h + \frac{1}{2}\frac{\partial^2\phi}{\partial x^2}(x_i, y_j)h^2 \pm \frac{1}{3!}\frac{\partial^3\phi}{\partial x^3}(x_i, y_j)h^3 + O(h^4),$$
$$\phi(x_i, y_j \pm h) = \phi(x_i, y_j) \pm \frac{\partial\phi}{\partial y}(x_i, y_j)h + \frac{1}{2}\frac{\partial^2\phi}{\partial y^2}(x_i, y_j)h^2 \pm \frac{1}{3!}\frac{\partial^3\phi}{\partial y^3}(x_i, y_j)h^3 + O(h^4).$$

$\phi_{ij} = \phi(x_i, y_j)$ として辺々を加えると

$$\phi_{i+1,j} + \phi_{i-1,j} = 2\phi_{i,j} + \frac{\partial^2\phi}{\partial x^2}(x_i, y_j)h^2 + O(h^4),$$
$$\phi_{i,j+1} + \phi_{i,j-1} = 2\phi_{i,j} + \frac{\partial^2\phi}{\partial y^2}(x_i, y_j)h^2 + O(h^4).$$

この辺々を加えて整理すると 2 次元の laplacian の差分近似を得る．

$$\triangle\phi(x_i, y_j) \approx \frac{\phi_{i+1,j} + \phi_{i-1,j} + \phi_{i,j+1} + \phi_{i,j-1} - 4\phi_{i,j}}{h^2} \tag{9}$$

この離散化誤差は $O(h^2)$ である．近似 (9) による (8) の数値計算では，計算したい点 (x_i, y_j) および隣接する 4 つの格子点 $(x_{i+1}, y_j), (x_{i-1}, y_j), (x_i, y_{j+1}), (x_i, y_{j-1})$ での ϕ, ψ の値が必要である．

4 粘弾性体の表面応力

境界 $\partial\Omega$ における (4) の意味について考える．本節では，再び $\phi = (\phi_1, \phi_2, \phi_3), \psi = (\psi_1, \psi_2, \psi_3)$ とする．

まず，式 (Y20') の flux は次のとおり．

$$f_i = \mu \sum_j \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) n_j + \lambda(\nabla \cdot u)n_i + \eta \sum_j \left(\frac{\partial \dot{u}_i}{\partial x_j} + \frac{\partial \dot{u}_j}{\partial x_i}\right) n_j + \tilde{\zeta}(\nabla \cdot \dot{u})n_i. \tag{10}$$

表面応力が $f_i = f_i^{(c)}(x)\cos\omega t + f_i^{(s)}(x)\sin\omega t$ で与えられるなら，(Y22') を考えて次を得る．

$$f_i^{(c)} = \mu \sum_j \left(\frac{\partial \phi_i}{\partial x_j} + \frac{\partial \phi_j}{\partial x_i}\right) n_j + \lambda(\nabla \cdot \phi)n_i + \omega\eta \sum_j \left(\frac{\partial \psi_i}{\partial x_j} + \frac{\partial \psi_j}{\partial x_i}\right) n_j + \omega\tilde{\zeta}(\nabla \cdot \psi)n_i, \tag{11a}$$

$$f_i^{(s)} = \mu \sum_j \left(\frac{\partial \psi_i}{\partial x_j} + \frac{\partial \psi_j}{\partial x_i}\right) n_j + \lambda(\nabla \cdot \psi)n_i - \omega\eta \sum_j \left(\frac{\partial \phi_i}{\partial x_j} + \frac{\partial \phi_j}{\partial x_i}\right) n_j - \omega\tilde{\zeta}(\nabla \cdot \phi)n_i \tag{11b}$$

式 (Y31) と組み合わせて考える表面応力は (11) が自然である．

表面応力を ϕ と ψ にわけて考えられるか?

(11) に物理的な意味はつくか? (10) は意味付けられそうである．実際，通常の等方弾性体方程式では，変位 $\tilde{u} = (\tilde{u}_1, \tilde{u}_2, \tilde{u}_3)$ に対して表面応力の i 方向の成分 $\tilde{f}_i$ は

$$\tilde{f}_i = \mu \sum_j \left(\frac{\partial \tilde{u}_i}{\partial x_j} + \frac{\partial \tilde{u}_j}{\partial x_i} \right) n_j + \lambda (\nabla \cdot \tilde{u}) n_i$$

で与えられる．

式 (Y31) を近似して得られる (Y44) (あるいは (1)) に対する境界条件を考える．そのためには (11) を近似すればよい．例えば非圧縮として $\nabla \cdot \phi = \nabla \cdot \psi = 0$ を仮定すると次を得る．

$$f_i^{(c)} = \mu \frac{\partial \phi_i}{\partial n} + \mu \sum_j \frac{\partial \phi_j}{\partial x_i} n_j + \omega\eta \frac{\partial \psi_i}{\partial n} + \omega\eta \sum_j \frac{\partial \psi_j}{\partial x_i} n_j, \tag{12a}$$

$$f_i^{(s)} = \mu \frac{\partial \psi_i}{\partial n} + \mu \sum_j \frac{\partial \psi_j}{\partial x_i} n_j - \omega\eta \frac{\partial \phi_i}{\partial n} - \omega\eta \sum_j \frac{\partial \phi_j}{\partial x_i} n_j. \tag{12b}$$

従って

$$\mu \sum_j \frac{\partial \phi_j}{\partial x_i} n_j + \omega\eta \sum_j \frac{\partial \psi_j}{\partial x_i} n_j \text{ が } \mu \frac{\partial \phi_i}{\partial n} + \omega\eta \frac{\partial \psi_i}{\partial n} \text{ に比べて無視できる大きさであり，} \tag{13a}$$

$$\mu \sum_j \frac{\partial \phi_j}{\partial x_i} n_j + \omega\eta \sum_j \frac{\partial \psi_j}{\partial x_i} n_j \text{ が } \mu \frac{\partial \phi_i}{\partial n} + \omega\eta \frac{\partial \psi_i}{\partial n} \text{ に比べて無視できる大きさである，} \tag{13b}$$

と仮定してよければ，

$$\mu \sum_j \frac{\partial \phi_j}{\partial x_i} n_j + \omega\eta \sum_j \frac{\partial \psi_j}{\partial x_i} n_j = \mu \sum_j \frac{\partial \phi_j}{\partial x_i} n_j + \omega\eta \sum_j \frac{\partial \psi_j}{\partial x_i} n_j = 0 \tag{14}$$

と近似される．したがって (4) が表面応力を表すと考えることができて境界条件 (5d) は妥当であるといえる．

仮定 (14) は，次のように表せる．

$$\left(\mu \frac{\partial \phi}{\partial x_i} + \omega\eta \frac{\partial \psi}{\partial x_i} \right) \cdot n = 0 \quad \text{かつ} \quad \left(\mu \frac{\partial \psi}{\partial x_i} - \omega\eta \frac{\partial \phi}{\partial x_i} \right) \cdot n = 0$$

$$\Leftrightarrow \quad \mu \sum_j \frac{\partial \phi_j}{\partial x_i} n_j + \omega\eta \sum_j \frac{\partial \psi_j}{\partial x_i} n_j = 0 \quad \text{かつ} \quad \mu \sum_j \frac{\partial \psi_j}{\partial x_i} n_j - \omega\eta \sum_j \frac{\partial \phi_j}{\partial x_i} n_j = 0.$$

$$\Leftrightarrow \quad \sum_j \frac{\partial \phi_j}{\partial x_i} n_j = \sum_j \frac{\partial \phi_j}{\partial x_i} n_j = 0 \quad \left(\frac{\partial \phi}{\partial x_i} \cdot n = \frac{\partial \psi}{\partial x_i} \cdot n = 0 \right).$$

この仮定の物理的な意味は? この仮定 (13) は，仮定 (Y45) と大きく関係していると考えられる．仮定 (Y34)(Y43)(Y45) から導出できるか?

5 考えるべきこと

- モデルの仮定をどこまで緩めるか．どのモデルを出発点とするか．
- P 波 (疎密波) のとき，どのようにモデル化するか．

- MRE で測定できる物理量は何か．ϕ, ψ を求めることは可能か?
- 係数 η, μ が定数でなく，区分的に滑らか (微分可能) な場合を想定し，(1) から係数同定の式を導けるか．平均値 (期待値) の形式でもよい．